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ON THE CALCULATION OF STEADY BOUNDARY LAYERS FOR CONTINUOUS SUCTION, WITH DISCONTINUOUSLY VARIABLE SUCTION VELOCITY

By Werner Rheinboldt

Translation of "Zur Berechnung stationärer Grenzschichten bei kontinuierlicher Absaugung mit unstetig veränderlicher Absaugegeschwindigkeit." Inaugural dissertation for obtaining the degree of Doctor of the Faculty of Natural Sciences and Mathematics of the Albert-Ludwigs-University at Freiburg im Breisgau, 1955.

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ON THE CALCULATION OF STEADY BOUNDARY LAYERS FOR CONTINUOUS SUCTION, WITH DISCONTINUOUSLY VARIABLE SUCTION VELOCITY*

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SUMMARY

Almost all solutions, so far known, of the problem of exact calculation of the velocity distribution in a boundary layer under the influence of continuous suction pertain to the class of "similar" solutions. One deals, therefore, with individual particular integrals of the boundary-layer equations under special boundary conditions. Compilations may be found, for instance, in H. Schlichting [1] or E. J. Watson [2].

If one disregards the reports using the so-called Pohlhausen methods, thus not yielding rigorous solutions of the boundary-layer equations, there exist only very few investigations which deal with the suction boundary layer for arbitrarily prescribable boundary conditions. Here belongs, for instance, the paper by R. Iglisch [3] which treats the onset of the boundary-layer flow on a flat plate in longitudinal flow with homogeneous suction. On the other hand, the case of merely piecewise suction for otherwise impermeable wall - which is of extreme interest for practical cases - has so far not been rigorously investigated. The main reason probably is that at the beginning and at the end of every suction region the value of the v velocity component at the wall becomes discontinuous so that all customary calculation methods fail there.

In the present report, we shall develop a method, on the example of a jumplike start of suction for arbitrary external pressure distribution and arbitrary suction law - a method which permits the exact calculation of the rapid variations of the velocity distribution (according

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to boundary-layer theory) near arbitrary flow discontinuities of the kind mentioned. It is assumed that one deals with a laminar, two-dimensional, steady boundary layer of an incompressible fluid. Farther downstream where the variations of the velocity in the direction of the main flow are no longer so large, one may again use one of the customary continuation methods.

The method used consists, essentially, in setting up a series expansion for the stream function, after an appropriate transformation of variables. For larger distances from the wall, an asymptotic expansion is then connected to that series which is usable only in the proximity of the wall.

The theory of the method is contained in chapters 2 to 5. In chapter 6, all formulas necessary for the practical application are compiled. In chapter 7 there follow a few examples showing the usefulness of the method.

The present report was suggested by Professor Dr. Görtler. I want to express to Professor Görtler my deep gratitude for many fruitful discussions and for his great interest in the progress of the work. Also, I should like to thank Miss Herlinde Kompe for her help in the performance of the numerical calculations.

1. STATEMENT OF THE PROBLEM

If a fluid with flight friction (laminar) flows around a body, the flow may be regarded as frictionless, outside a zone near the wall - the boundary layer. For calculation of the velocity distribution within the boundary layer, the Navier-Stokes equations may be replaced by the simpler boundary-layer equations, according to Prandtl [4]. For the case of two-dimensional steady flow of incompressible fluids, these equations read

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad (1)$$

where

x signifies the arc length of the wall in the direction of the flow

y	the perpendicular distance from the wall
$u = u(x,y)$	the velocity component in the x-direction
$v = v(x,y)$	the velocity component in the y-direction
ρ	the density (constant)
ν	the kinematic viscosity (constant)

The pressure $p(x)$ can be determined from the frictionless outer flow to be regarded as known.[1] If $u_\infty(x)$ denotes the longitudinal velocity at the edge of the boundary layer, there applies in an approximation according to boundary-layer theory

$$-\frac{1}{\rho}\frac{dp}{dx} = u_\infty(x)\,\frac{du_\infty(x)}{dx} \tag{2}$$

If the boundary layer is now sucked off entirely or partly through the wall of the body, the boundary-layer approximations performed in the Navier-Stokes equations are known to remain justified as long as the ratio of the speed of suction to the outer flow $u_\infty(x)$ is sufficiently small. If, however, this ratio becomes too large, a dependence of the boundary-layer pressure p on the transverse coordinate y appears. This is the so-called sink effect. For what follows, we shall always assume the speed of suction to be so small that the boundary-layer equations (1) and the pressure equation (2) remain valid.

Various boundary conditions are now added to the boundary-layer equations. Due to the adherence of the fluid to the wall, there is, first of all

$$u(x,0) = 0 \tag{3}$$

Furthermore, the transition of the boundary-layer flow $u(x,y)$ to the outer flow $u_\infty(x)$ is regulated by the requirement

$$\lim_{y \to \infty} u(x,y) = u_\infty(x) \tag{4}$$

Finally, for continuous suction

$$v(x,0) = -v_0(x) \tag{5}$$

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[1]In practice, this is done mostly by measurement of the pressure.

is valid where $v_0(x)$ represents the suction distribution ($v_0(x) > 0$ suction, $v_0(x) < 0$ blowing off).[2] Especially for $v_0(x) \equiv 0$, thus

$$v(x,0) = 0 \tag{6}$$

(5) gives the case of the impermeable wall.

As shown in the comprehensive report of H. G. Lew - R. D. Mathieu [5], in almost all (rigorous) theoretical investigations concerning continuous boundary-layer suction up till now the restrictive assumption was made that suction prevails everywhere along the body in the flow, and that the suction distribution $v_0(x)$ is continuous. On the other hand, the more comprehensive case is of considerably higher practical interest where the wall is piecewise alternately impermeable as well as porous; thus the function $v_0(x)$ identically disappears within certain x-intervals and becomes discontinuous at the beginning and end of every region of suction in general. For instance, most experimental reports dealing with this field use such more general suction distributions. Compare for instance, the reports of Sir Jones [6], W. Pfenninger [7], and A. v. Doenhoff - L. K. Loftin [8].

In the present report we shall treat the exact calculation of the boundary-layer flow for such suction distributions which are discontinuous in places. We may limit ourselves to the special case that the wall, starting from its beginning ($x = 0$) is impermeable at first, until the continuous suction begins abruptly at an arbitrary point $x = x_0 > 0$. Accordingly, we have to use, in addition to (3) and (4), as a third boundary condition

$$v(x,0) = \left\{ \begin{array}{lll} 0 & \text{for } 0 \leqq x < x_0 & \\ -v_0(x) & \text{for } x_0 \leqq x & (v_0(x_0) \neq 0) \end{array} \right\} \tag{7}$$

We shall see later that, with the aid of the method developed for the solution of this case, the problem of an abrupt stopping of suction also can be solved, likewise any arbitrary discontinuous variation of the suction distribution $v_0(x)$, so that our above restriction to the beginning of suction is by no means essential.

It is well known that the three boundary conditions (3), (4), and (7) are not sufficient to determine the boundary-layer flow for all

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[2] The boundary conditions (3) and (5) together correspond to a perpendicular continuous suction which can be realized technically with the aid of porous walls, made of sintered bronze, for instance.

$x \geqq 0$ completely. For this, one rather needs, corresponding to the parabolic character of the boundary-layer equations, in addition, an initial condition for $u(0,y)$ in $x = 0$.

In the interval $0 \leqq x \leqq x_0$, our initial boundary-value problem defined by the equations (1), (2), and the secondary conditions (3), (4), (7) as well as by the initial condition $u(0,y)$ represents precisely the usual boundary-layer problem without suction. Thus, we may assume directly, with a view to the aim of our statement of the problem, that in this range the desired solution has already been calculated with the aid of the known methods, for instance, the Blasius series or one of the customary continuation methods. Then we know also, at the point $x = x_0$, the velocity distribution $u(x_0,y) = \tilde{u}(y)$. This function $\tilde{u}(y)$ represents, because of the boundary condition (7), the end profile of the boundary layer without suction and contains the entire previous history of the flow up to the point $x = x_0$. For the further calculation of the boundary layer in the region $x \geqq x_0$, one may now use, instead of the initial condition for $u(0,y)$, simply

$$u(x_0,y) = \tilde{u}(y) \tag{8}$$

as a new initial condition.

With the aid of the entrance profile $\tilde{u}(y)$, there follows from the boundary-layer equations (1) and (2)

$$u\ u_x + v\ u_y = -u\ v_y + v\ u_y = -u^2 \frac{\partial}{\partial y}\left(\frac{v}{u}\right) = u_\infty u_\infty' + \nu\ u_{yy} \tag{9}$$

thus

$$v(x_0,y) = \tilde{v}(y) = -\tilde{u} \int_0^y \frac{\nu\ \tilde{u}''(y) + u_\infty(x_0)u_\infty'(x_0)}{(\tilde{u}(y))^2}\, dy \tag{10}$$

and, as may be easily confirmed

$$v(x_0,0) = \tilde{v}(0) = 0 \tag{11}$$

is valid. The discontinuity caused by the jump-type start of suction lies therefore at the point $x = x_0$, $y = 0$ at the transition from the boundary v values $v(x,0) = -v_0(x)$ to $y = 0$ and of the initial v values $v(x_0,y) = \tilde{v}(y)$ to $x = x_0$.

We want to remark here briefly that the initial condition (8) and the outer boundary condition (4) are by no means independent of one another. As the author has shown in another report [9], there applies for every two-dimensional, steady, and incompressible boundary-layer problem the following theorem, independently of the form of the inner boundary conditions under certain assumptions not restrictive for the present case. If at any point $x = x_1$ an initial condition fitting the problem $u(x_1,y) = \tilde{u}(y)$ has been set up and the occurring entrance profile $\tilde{u}(y)$ correctly adjoins the outer flow $u_\infty(x)$ - that is, $\lim\limits_{y \to \infty} \tilde{u}(y) = u_\infty(x_1)$ - the outer boundary condition (4) has already been automatically satisfied in an interval $x_1 \leqq x \leqq x_2$.

It is expedient for what follows to write the boundary-layer equations in dimensionless form. If L signifies a characteristic length, U a characteristic velocity, and $Re = \frac{UL}{\nu}$ the pertaining Reynolds number, we put

$$\left.\begin{aligned} x^* &= \frac{x}{L} & u^* &= \frac{u}{U} & u_\infty{}^* &= \frac{u_\infty}{U} & \tilde{u}^* &= \frac{\tilde{u}}{U} \\ y^* &= \frac{y}{L}\sqrt{Re} & v^* &= \frac{v}{U}\sqrt{Re} & v_0{}^* &= \frac{v}{U}\sqrt{Re} & p^* &= \frac{p}{\rho U^2} \end{aligned}\right\} \quad (12)$$

Since only this coordinate system is being used below, up to chapter 7, we may omit there the asterisks without having to be afraid of confusion.

The boundary-layer equations (1), (2) read, with use of these new quantities

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= u_\infty u_\infty{}' + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \quad (13)$$

and the boundary conditions become

$$u(x,0) = 0 \qquad v(x,0) = -v_0(x) \qquad (x \geqq x_0) \tag{14a}$$

$$\lim_{y \to \infty} u(x,y) = u_\infty(x) \tag{14b}$$

$$u(x_0,y) = \tilde{u}(y) \tag{14c}$$

In the present report, we intend to develop a method which permits rigorous calculation of the rapid variations in the velocity distribution in the boundary layer, in the proximity of the flow discontinuity. Farther downstream from the point $x = x_0$, where the changes in the velocity distribution are no longer as large, we may use again one of the customary step-by-step methods like, for instance, the difference method developed by H. Görtler [10]. A slight improvement of this difference method by Görtler, which proved very good in such calculations with suction, is described in the appendix.

Finally, it must be remarked that the jump of the boundary v-values at the point $x = x_0$, $y = 0$ is propagated into the flow along the characteristics $x = x_0$, $y > 0$ as a discontinuity of certain higher derivatives of the solution. We shall not discuss this fact in more detail since it is not necessary for our further considerations. Its qualitative correctness is confirmed, for instance, by the results of our examples. (Cf., for instance, also Courant-Hilbert [11], vol. II, p. 299.) The real physical flow has, of course, no such discontinuity on $x = x_0$. One deals here solely with a local degeneration phenomenon of mathematical type which is caused by the boundary-layer approximations. Actually - in agreement with the elliptical character of the complete Navier-Stokes equations - no discontinuous jump takes place on $x = x_0$ but a continuous, though rapid, transition.

2. STATEMENT OF THE NEW METHOD OF SOLUTION

For the initial boundary-value problem defined by the equations (13) and (14), we shall assume below that the three given functions $\tilde{u}(y)$, $u_\infty(x)$, and $v_0(x)$ are analytical, thus may be represented by uniformly convergent power series

$$\tilde{u}(y) = \sum_{n=1}^{\infty} a_n y^n \quad [3] \qquad (15)$$

$$u_\infty(x) = \sum_{n=0}^{\infty} u_n (x - x_0)^n \qquad (16)$$

$$v_0(x) = \sum_{n=0}^{\infty} v_n (x - x_0)^n \qquad (v_0 \neq 0) \qquad (17)$$

[3] Note that $\tilde{u}(0) = 0$, because of $u(x,0) = 0$.

R. Iglisch [3] eliminated a difficulty in the calculation of the approach flow - similar to the discontinuity of the boundary v-vlaues at the point $x = x_0$, $y = 0$ (cf. chapter 1) - at the leading edge of the plane plate in longitudinal flow with homogeneous suction. He maps, essentially, the straight line $x = 0$, thus the carrier of the flow discontinuity, into the infinitely distant point $\overline{x} = 0$, $\overline{y} = \infty$, with the aid of a coordinate transformation of the form $\overline{x} = x$, $\overline{y} = \frac{y}{\sqrt{x}}$. It suggests the use of a similar procedure here. However, we shall see that in our case a transformation of the same kind is not sufficient, and we use therefore directly the more general transformation

$$\left.\begin{aligned} \sigma &= \sqrt[N]{x - x_0} \\ \tau &= \frac{y}{N\sqrt[N]{x - x_0}} \\ \psi(x,y) &= \sigma^{N-1}\, f(\sigma,\tau) \end{aligned}\right\} \tag{18}$$

which was applied first by S. Goldstein [12] in the calculation of the wake behind the plane plate. N is an integer still to be defined, and $\psi(x,y)$ is the stream function of our problem for which, therefore

$$\psi_y(x,y) = u(x,y) \qquad\qquad \psi_x(x,y) = -v(x,y) \tag{19}$$

is valid. One then obtains

$$u = \frac{\sigma^{N-2}}{N}\, f \tag{20}$$

and, as can be easily checked, the differential equations (13) are transformed into

$$N^3\sigma^{4-N}u_\infty u_\infty' + f_{\tau\tau\tau} - (N - 2)f_\tau^{\,2} + (N - 1)f\, f_{\tau\tau} - \sigma f_\tau f_{\tau\sigma} + \sigma f_\sigma f_{\tau\tau} = 0 \tag{21}$$

If we introduce the abbreviation

$$C_0(x) = \int_{x_0}^{x} v_0(x)\, dx = \sum_{n=1}^{\infty} \frac{v_{n-1}}{n}\,(x - x_0)^n \tag{22}$$

as a dimensionless mass coefficient,[4] we obtain from (14a) and (14b) as new boundary conditions

$$f_\tau(\sigma,0) = 0 \qquad f(\sigma,0) = \sigma^{1-N}\, C_0(\sigma^N) = \sum_{n=0}^{\infty} \frac{v_n}{n+1}\, \sigma^{Nn+1} \quad ^{5} \tag{23a}$$

and

$$\frac{\sigma^{N-2}}{N}\, f_\tau(\sigma,\infty) = u_\infty(\sigma^N) = \sum_{n=0}^{\infty} u_n \sigma^{Nn} \tag{23b}$$

Since in the transformation (18) the semi-infinite line $x = x_0$, $y > 0$ is transformed into the one (infinitely distant) point $\sigma = 0$, $\tau = \infty$, the initial condition (14c), the entrance profile $\tilde{u}(y)$ which is so important for our problem, is at first completely lost.

We shall show first that for $N = 2$, that is, for the case of the transformation used by Iglisch, there results no possibility of taking the entrance profile again into consideration. For $N = 2$, the differential equation (21) assumes the form

$$8\sigma^2 u_\infty u_\infty' + f_{\tau\tau\tau} + f\, f_{\tau\tau} - \sigma f_\tau f_{\tau\sigma} + \sigma f_\sigma f_{\tau\tau} = 0$$

and the boundary conditions (23a, 23b) become

$$f_\tau(\sigma,0) = 0 \qquad f(\sigma,0) = \frac{C_0(\sigma^2)}{\sigma} = \sum_{n=0}^{\infty} \frac{v_n}{n+1}\, \sigma^{2n+1} \tag{24a}$$

$$f_\tau(\sigma,\infty) = 2u_\infty(\sigma^2) = 2\sum_{n=0}^{\infty} u_n \sigma^{2n} \tag{24b}$$

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[4]Thus, the suction quantity in the interval from x_0 to x, when b is the width of the suction region traversed by the flow, amounts in our dimensionless coordinates to

$$Q = bC_0(x)$$

[5]Here we put tacitly $f(0,0) = 0$ since an additive constant is unessential in the stream function.

In spite of the lacking initial condition, the solution $f(o,\tau)$ of (24a/b) is fully determined in the neighborhood of $\sigma = 0$. If one states for it a power-series expression

$$f(\sigma,\tau) = \sum_{n=0}^{\infty} f_n(\tau)\sigma^n \qquad (25)$$

there result for the coefficients $f_n(\tau)$ in every case differential equations of the third order with boundary conditions for $f(0)$, $f'(0)$, and for $f'(\infty)$. It can be proved - which we shall not do, however - that thereby all $f_n(\tau)$ are completely determined. For $f_0(\tau)$, for instance, the differential equation reads

$$f_0''' + f_0 f_0'' = 0$$

and the boundary conditions have the form

$$f_0(0) = f_0'(0) = 0 \qquad f_0'(\infty) = \text{const}$$

$f_0(\tau)$ is therefore exactly equal to Blasius' plate profile. Thus, the lacking initial condition must be necessarily

$$f(0,\tau) = f_0(\tau)$$

and one can now calculate $f(\sigma,\tau)$ immediately, for instance - as Iglisch did - with the aid of a numerical step-by-step method instead of by means of the series (25). However, one understands at once that the solution $u(x,y)$, $v(x,y)$ of (13) obtained from this $f(\sigma,\tau)$ by the inverse transformation (18), satisfies only a constant entrance profile $u(0,y) = \text{const}$. Thereby, we have merely regained Iglisch's solution, however, for arbitrary external pressure distribution and without the auxiliary transformations additionally used by Iglisch.

We now set $N = 3$. Then the differential equation (21) reads

$$27\sigma u_\infty u_\infty' + f_{\tau\tau\tau} - f_\tau^{\,2} + 2f\, f_{\tau\tau} - \sigma f_\tau f_{\tau\sigma} + \sigma f_\sigma f_{\tau\tau} = 0 \qquad (26)$$

and the boundary conditions (23 a/b) are transformed into

$$f_\tau(\sigma,0) = 0 \qquad f(\sigma,0) = \sigma^{-2}\, C_0(\sigma^3) = \sum_{n=0}^{\infty} \frac{v_n}{n+1}\, \sigma^{3n+1} \qquad (27a)$$

$$\sigma f_\tau(\sigma,\infty) = 3u_\infty(\sigma^3) = 3 \sum_{n=0}^{\infty} u_n \sigma^{3n} \tag{27b}$$

Here, $f(\sigma,\tau)$ is no longer fully determined in the neighborhood of $\sigma = 0$. If (in order to understand this) one again sets up a power-series expression of the form (25), there results, for instance, for $f_0(\tau)$ the differential equation

$$f_0''' + f_0 f_0'' - f_0'^2 = 0$$

with the boundary conditions

$$f_0(0) = f_0'(0) = 0$$

The outer boundary condition (27b), in contrast, loses its significance and must be cancelled. Then, $f_0(\tau)$ is no longer uniquely determined, however. The same is true for the other coefficients $f_n(\tau)$. Everytime, one boundary condition is lacking in their differential equations and we are free to choose it appropriately. Of course, we shall attempt, through this choice, to take the entrance profile $\tilde{u}(y)$ into account, after all, as S. Goldstein did for the wake behind the plane plate. For this purpose, we put temporarily $\sigma = \frac{y}{3\tau}$, thus pass over to a τ, y-coordinate system. Then there follows from (20) and (25)

$$u = \sum_{n=1}^{\infty} \frac{f'_{n-1}(\tau)}{3^{n+1}\,\tau^n}\, y^n$$

Since for fixed y the limiting process $x \to 0$ in the τ, y-coordinate system signifies $\tau \to \infty$, (14c) and (15) are transformed into

$$\lim_{\tau\to\infty} \sum_{n=1}^{\infty} \frac{f'_{n-1}(\tau)}{3^{n+1}\,\tau^n}\, y^n = \sum_{n=1}^{\infty} a_n y^n$$

this results in the case of uniform convergence of the series (which is to be assumed)

$$\lim_{\tau\to\infty} \frac{f'_{n-1}(\tau)}{\tau^n} = 3^{n+1} a_n \qquad (n=1,2,\ldots) \tag{28}$$

This relation may be used directly as outer boundary condition for the $f_n(\tau)$.

The elimination of the former outer boundary condition $u(x,\infty) = u_\infty(x)$ is unimportant for the further calculation. As has already been stated in section 1, this boundary condition is, under certain assumptions, always automatically satisfied in an interval $x_0 \leqq x \leqq x_1$, if only the flow corresponds to the entrance profile $\tilde{u}(y)$ - which is attained by (28) - and if

$$\lim_{y \to \infty} \tilde{u}(y) = u_\infty(x_0) \tag{29}$$

is valid for this entrance profile - and this, of course, must be the case, if only because of the previous history of the flow. We shall discuss this point in more detail in section 5.

In order to find the differential equations for the coefficients $f_n(\tau)$ individually, we have to enter into the differential equation (26) with the expressions (16) and (25). If we write abbreviately

$$p_n = \sum_{m=0}^{n} (m + 1)u_{m+1}u_{n-m} \tag{30}$$

thus

$$u_\infty u_\infty' = \sum_{n=0}^{\infty} p_n \sigma^{3n} \tag{31}$$

there results after comparison of the coefficients

$$\left.\begin{aligned} & f_0''' + 2f_0f_0'' - f_0'^2 = 0 \\ & f_n''' + 2f_0f_0'' - (n + 2)f_0'f_n' + (n + 2)f_0''f_n = F_n(\tau) \qquad (n=1,2,\ldots) \end{aligned}\right\} \tag{32}$$

with

$$F_n(\tau) = \sum_{m=1}^{n-1} \left[(m + 1)f_m'f_{n-m}' - (m + 2)f_mf_{n-m}''\right] - \begin{cases} 27p_j & \text{for } n = 3j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

This infinite system of ordinary differential equations may be solved by recurrence methods.

As boundary conditions, we obtain from (27a) and (28) for $n = 0, 1, 2, \ldots$

$$f_n(0) = \begin{cases} v_j & \text{for} \quad n = 3j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{34a}$$

$$f_n{}'(0) = 0 \tag{34b}$$

$$\lim_{\tau \to \infty} \frac{f_n{}'(\tau)}{\tau^{n+1}} = 3^{n+2}\, a_{n+1} \tag{34c}$$

The first equation (32) may be integrated immediately. One finds as the solution to the boundary conditions (34)

$$f_0(\tau) = \frac{9}{2}\, a_1 \tau^2 \tag{35}$$

If one introduces this into the other equations (32), it is expedient to choose at the same time as a new variable

$$\eta = \sqrt[3]{9a_1}\; \tau \qquad f_n(\tau) \equiv g_n(\eta) \qquad (n=1,2,\ldots) \tag{36}$$

Thus, we obtain for $n = 1, 2 \ldots$ the sequence of linear differential equations

$$g_n{}''' + \eta^2 g_n{}'' - (n + 2)\eta\, g_n{}' + (n + 2)g_n = G_n(\eta) \tag{37}$$

with

$$G_n(\eta) = \frac{1}{\sqrt[3]{9a_1}} \sum_{m=1}^{n-1} \left[(m + 1)g_m{}'g'_{n-m} - (m + 2)g_m g''_{n-m}\right] - \begin{cases} 27p_j & \text{for} \quad n = 3j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

and the boundary conditions

$$g_n(0) = \begin{cases} v_j & \text{for} \quad n = 3j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{39a}$$

$$g_n'(0) = 0 \tag{39b}$$

$$\lim_{\eta \to \infty} \frac{g_n'(\eta)}{\eta^{n+1}} = \left[\frac{3}{\sqrt[3]{9a_1}}\right]^{n+2} a_{n+1} \tag{39c}$$

3. ON THE SOLUTIONS OF THE HOMOGENEOUS EQUATIONS (37)

It is important for what follows, to know the full diversity of solutions of the homogeneous equations

$$g_n''' + \eta^2 g_n'' - \eta(n + 2)g_n' + (n + 2)g_n = 0 \qquad (n=1,2,\ldots) \tag{40}$$

pertaining to the differential equations (37), and also to be able to master the asymptotic behavior of these solutions for large η.

One sees immediately that $g_n = \eta$ always must be a solution of (40). Thus, (40) may be reduced, by the expression

$$g_n = \eta \int \overline{g}_n(\eta)\, d\eta$$

to the differential equation of the second order

$$\eta\, \overline{g}_n''(\eta) + (3 + \eta^2)\overline{g}_n'(\eta) - n\, \eta^2 \overline{g}_n(\eta) = 0 \tag{41}$$

for a new desired function $\overline{g}_n(\eta)$. Equation (41), in turn, is transformed into the differential equation of the confluent hypergeometrical functions

$$\xi\, w''(\xi) + (b - \xi)w'(\xi) - a\, w(\xi) = 0 \tag{42}$$

if one puts

$$\xi = -\frac{\eta^3}{3}, \quad w = \overline{g}_n \quad \text{and} \quad a = -\frac{n}{3}, \quad b = \frac{5}{3}$$

If, therefore, $w(\xi)$ is a solution of (42) with $a = -\frac{n}{3}$, $b = \frac{5}{3}$, the equation $\bar{g}_n(\eta) = w\left(-\frac{\eta^3}{3}\right)$ represents a solution of (41).

The differential equation (42) is known to have the two linearly independent solutions.

$$w_1(\xi) = {}_1F_1(a,b,\xi) \equiv \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\xi^n}{n!}$$

$$w_2(\xi) = \xi^{1-b}{}_1F_1(a - b + 1; 2 - b; \xi)$$

where for the sake of brevity we wrote $(a)_n = a(a + 1) \ . \ . \ . \ (a + n - 1)$, $(a)_0 = 1$. Thereby

$$h_n(\eta) = \eta \int {}_1F_1\left(-\frac{n}{3}, \frac{5}{3}, -\frac{\eta^3}{3}\right) d\eta$$

$$k_n(\ ) = \eta \int \eta^{-2}{}_1F_1\left(-\frac{n + 2}{3}, \frac{1}{3}, -\frac{\eta^3}{3}\right) d\eta \tag{43}$$

$$= 1 + \eta \int \eta^{-2}\left[{}_1F_1\left(-\frac{n + 2}{3}, \frac{1}{3}, -\frac{\eta^3}{3}\right) - 1\right] d\eta$$

are two solutions of (40). These solutions are both fully analytical functions of η. In $h_n(\eta)$ this can be seen immediately since ${}_1F_1(a,b,\xi)$ is known to have this property. For $k_n(\eta)$ it follows from the second representation since here also the integrand is fully analytical. Moreover, η, $h_n(\eta)$, $k_n(\eta)$, the three solutions obtained, are linearly independent of (40). This results immediately from the fact that the power-series expansion of $h_n(\eta)$ has the form

$$h_n(\eta) = \sum_{m=0}^{\infty} \frac{n(n - 3) \ \dots \ (n - 3(m - 1))}{5\cdot 8\cdot 11 \ \dots \ (5 + 3(m - 1))} \frac{\eta^{3m+2}}{3^m m!(3m + 1)} \tag{44}$$

thus begins with η^2 whereas that of $k_n(\eta)$ starts with 1, as may easily be checked.

In order to find the asymptotic expansions of the functions $h_n(\eta)$ and $k_n(\eta)$, one will first try to introduce for ${}_1F_1$ the well-known asymptotic expansion into (43) and then to integrate. In this manner, however, the constant terms of the desired expansions remain undetermined. Thus, we must choose another procedure. For this purpose, we shall represent the functions $h_n(\eta)$ and $k_n(\eta)$ first by certain integrals of the Mellin type which - as we shall see - may then be evaluated as desired.[6]

As one may confirm, for instance, from the formulas given by E. Whittaker and G. N. Watson [13], the representation

$$ {}_1F_1(a,b,\xi) = \frac{\Gamma(b)}{\Gamma(a)} \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{\Gamma(z)\Gamma(a - z)}{\Gamma(b - z)} \xi^{-z} \, dz \tag{45} $$

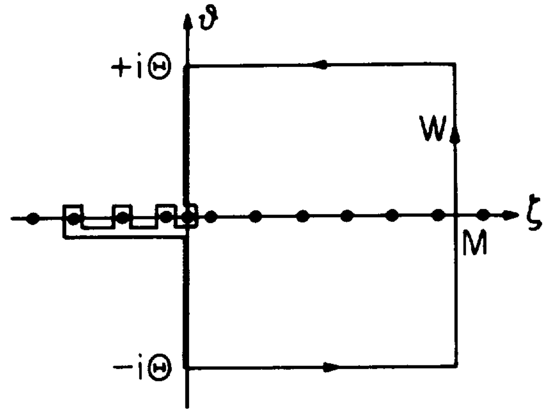
is valid for the confluent hypergeometrical function for $\underline{R}\xi > 0$ and arbitrary a, b which are not negative integers. The path of integration must separate the two pole chains stemming from $\Gamma(z)$ and $\Gamma(a - z)$; then the interval also converges uniformly. If one introduces this into (43) and interchanges the sequence of the integrations which is evidently permissible, one obtains the representations

$$ \left.\begin{aligned} h_n(\eta) &= \eta \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)} \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{\Gamma(z)\Gamma\left(-\frac{n}{3} - z\right)}{\Gamma\left(\frac{5}{3} - z\right)} \frac{\eta^{-3z+1}}{3^{-z}(-3z + 1)} \, dz \\ k_n(\eta) &= \eta \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(-\frac{n+2}{3}\right)} \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \frac{\Gamma(z)\Gamma\left(-\frac{n+2}{3} - z\right)}{\Gamma\left(\frac{1}{3} - z\right)} \frac{\eta^{-3z+1}}{3^{-z}(-3z + 1)} \, dz \end{aligned}\right\} \tag{46} $$

In order to derive from this the desired asymptotic expansions, the integrals are first calculated over a path W situated entirely in a finite domain. Their values can then be estimated when W makes piecewise the transition to infinity. The path W is to have, in the plane of $z = \zeta + i\vartheta$, the form drawn in the figure. * denotes poles of $\Gamma(z)$, 0 poles of $\Gamma\left(-\frac{n}{3} - z\right)$ or $\Gamma\left(-\frac{n+2}{3} - z\right)$, respectively. The point M

[6] This method is frequently used in the theory of the hypergeometrical functions.

is to have for the first integral the value $-\frac{n}{3} + m + \chi$ and for the second the value $-\frac{n+2}{3} + m + \chi (m = 2, 3, \ldots)$, with a suitable real number χ between zero and one.



The integrals (46), extended over W, may be immediately calculated according to the residue theorem: Inside W both integrands for $\underline{R}\eta > 0$ are certainly regular, except for the poles in

$$-\frac{n}{3},\ -\frac{n}{3}+1,\ \ldots,\ -\frac{n}{3}+m \text{ and } \frac{1}{3} \qquad (m=2,3,\ldots;\ n=1,2,\ldots)$$

or

$$-\frac{n+2}{3},\ -\frac{n+2}{3}+1,\ \ldots,\ -\frac{n+2}{3}+m \text{ and } -\frac{1}{3}$$

respectively.

For $n \neq 3j-1$ (j natural number), the pole at $1/3$ and $-1/3$ is different from the others and, like them, of the first order. Thus, one obtains in this case according to the residue theorem

$$\eta\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)}\frac{1}{2\pi i}\int_W \frac{\Gamma(z)\Gamma\left(-\frac{n}{3}-z\right)}{\Gamma\left(\frac{5}{3}-z\right)}\frac{\eta^{-3z+1}}{3^{-z}(-3z+1)}dz = \frac{\Gamma\left(\frac{5}{3}\right)}{3^{n/3}\Gamma\left(\frac{n+2}{3}\right)}\sum_{\nu=0}^{\infty}\frac{\left(-\frac{n}{3}\right)_\nu\left(-\frac{n+2}{3}\right)_\nu}{\nu!(n+1-3\nu)}3^\nu\eta^{n+2-3\nu} + \frac{3^{1/3}\Gamma\left(\frac{5}{3}\right)\Gamma\left(-\frac{n+1}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)}\eta \quad (47a)$$

or

$$\eta\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(-\frac{n+2}{3}\right)}\frac{1}{2\pi i}\int_W \frac{\Gamma(z)\Gamma\left(-\frac{n+2}{3}-z\right)}{\Gamma\left(\frac{1}{3}-z\right)}\frac{\eta^{-3z+1}}{3^{-z}(-3z+1)}dz = \frac{\Gamma\left(\frac{1}{3}\right)}{3^{\frac{n+2}{3}}\Gamma\left(\frac{n}{3}+1\right)}\sum_{\nu=0}^{m}\frac{\left(-\frac{n+2}{3}\right)_\nu\left(-\frac{n}{3}\right)_\nu}{\nu!(n+1-3\nu)}3^\nu\eta^{n+2-3\nu} - \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(-\frac{n+1}{3}\right)}{3^{1/3}\Gamma\left(-\frac{n+2}{3}\right)}\eta \quad (48a)$$

In the case $n = 3j - 1$, the pole at $-\frac{n}{3} + j$ $\left(\text{or } -\frac{n+2}{3} + j\right)$ coincides with the pole at $1/3$ (or $-1/3$). Thereby, the term with $\nu = j$ disappears in (47a) and (48a), and the additional term changes in (47a) into

$$\eta \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)} \operatorname*{Res}_{z = \frac{1}{3}} \frac{\Gamma(z)\Gamma\left(-\frac{n}{3} - z\right)\eta^{-3z+1}}{\Gamma\left(\frac{5}{3} - z\right)3^{-z}(-3z + 1)}$$

$$= \eta \frac{\Gamma\left(\frac{5}{3}\right)3^{1/3}(-1)^j}{\Gamma\left(\frac{1}{3} - j\right)\Gamma(j + 1)}\left[\ln \frac{\eta^3}{3} + \overline{\psi}(j + 1) - \overline{\psi}\left(\frac{1}{3}\right) - \overline{\psi}\left(\frac{4}{3}\right)\right] \tag{47b}$$

and in (48a) correspondingly into

$$-\eta \frac{\Gamma\left(\frac{1}{3}\right)(-1)^j}{3^{1/3}\Gamma\left(-\frac{1}{3} - j\right)\Gamma(j + 1)}\left[\ln \frac{\eta^3}{3} + \overline{\psi}(j + 1) - \overline{\psi}\left(\frac{2}{3}\right) - \overline{\psi}\left(\frac{1}{3}\right)\right] \tag{48b}$$

$\overline{\psi}$ here denotes the logarithmical derivative of the gamma function.

For abbreviation, we shall designate the expansion on the right side of (47a) which was broken off at $\nu = m$ by $\tilde{h}_{n,m}(\eta)$ and on the right side of (48a) by $\tilde{k}_{n,m}(\eta)$; the changes (47b) and (48b) will, of course, be taken into consideration for $n = 3j - 1$. Our assertion then is that $\tilde{h}_{n,\infty}(\eta)$ and $\tilde{k}_{n,\infty}(\eta)$ in a sector of the form

$$\Omega \qquad \underline{R}_\eta > 0 \qquad |\arg \eta| < \frac{\pi}{6} - \underline{\epsilon} \qquad (\underline{\epsilon} > 0 \text{ small})$$

are precisely the desired asymptotic expansions of $h_n(\eta)$ and $k_n(\eta)$ for $\eta \to \infty$. Since the proof for this is completely analogous for both functions, we limit ourselves to carrying it out for $h_n(\eta)$ only.

First, one may split up the integral on the left side of (47a) in the following manner:

$$\int_W = \underset{(\mathrm{I})}{\int_{-i\Theta}^{+i\Theta}} + \underset{(\mathrm{II})}{\int_{+i\Theta}^{-\frac{n}{3}+m+X+i\Theta}} - \underset{(\mathrm{III})}{\int_{-\frac{n}{3}+m+X-i\Theta}^{-\frac{n}{3}+m+X+i\Theta}} - \underset{(\mathrm{IV})}{\int_{-i\Theta}^{-\frac{n}{3}+m+X-i\Theta}} \tag{49}$$

We shall show that the two integrals (II) and (IV) for $\Theta \to \infty$ disappear for constant m and any η from the sector Ω. If we denote, abbreviately, the integrand by $I(z,\eta)$, thus

$$I(z,\eta) = \frac{\Gamma(z)\Gamma\left(-\frac{n}{3} - z\right)}{\Gamma\left(\frac{5}{3} - z\right)} \frac{\eta^{-3z+1}}{3^z(-3z + 1)}$$

we obtain, with use of Stirling's formula

$$I(z,\eta) = \sqrt{2\pi}\, e^{\left(z - \frac{1}{2}\right)\ln z} e^{-z} e^{\left(-\frac{n}{3} - \frac{5}{3}\right)\ln(-z)} 3^{-z} \frac{e^{(-3z+1)\ln \eta}}{(-3z + 1)} e^{\mu(z)}$$

with $\lim\limits_{|z| \to \infty} \mu(z) = 0$, therefore

$$\left|I(z,\eta)\right| = \sqrt{2\pi}\, e^{\left(\zeta - \frac{1}{2}\right)\ln|z| - \vartheta \arg z} e^{-\zeta} e^{\left(-\frac{n}{3} - \frac{5}{3}\right)\ln|z|} \times$$

$$e^{(-3\zeta+1)\ln|\eta| + 3\vartheta \arg \eta} \frac{e^{\zeta \ln 3}}{|-3z + 1|} e^{\underline{R}\mu(z)} \qquad (50)$$

For the logarithm, one must always take the main value, that is, $|\arg z| < \pi$ and $\vartheta \arg z$ is always positive. Thus, one obtains

$$|I(z,\eta)| \leqq \frac{\sqrt{2\pi}}{3\zeta + 1} |z|^{\zeta - \frac{n}{3} - \frac{5}{3} - \frac{1}{2}} e^{-\zeta(1 - \ln 3 + 3 \ln|\eta|) + \ln|\eta| + \underline{R}\mu(z)} \times$$

$$e^{-|\vartheta|\left(|\arg z| \pm 3 \arg \eta\right)}$$

with the upper sign being valid for $\vartheta < 0$, the lower for $\vartheta > 0$. Because of $|\arg z| \to \frac{\pi}{2}$ for $\Theta \to \infty$ on the integration sections of the integrals (II) and (IV) and because of $|\arg \eta| < \frac{\pi}{6} - \underline{\varepsilon}$, one can certainly find a $\Theta_0 > 0$, for any constant $\eta \epsilon \Omega$ for given $\delta > 0$, in such a manner that

$$0 < 2\delta < |\arg z| \pm 3 \arg \eta$$

for all $\Theta > \Theta_0$ and every sign is satisfied. Furthermore, there applies in the entire strip

$$0 < \zeta < -\frac{n}{3} + m + \chi, \quad \vartheta \text{ arbitrary}$$

for constant $\eta \epsilon \Omega$ evidently the estimates

$$\frac{\sqrt{2\pi}}{3\zeta + 1} e^{-\zeta(1 - \ln 3 + 3 \ln \eta) + \ln|\eta| + \underline{R}\mu(z)} < C_1$$

and

$$|z|^{\zeta - \frac{n}{3} - \frac{5}{3} - \frac{1}{2}} < C_2 |\vartheta|^{-\frac{n}{3} - \frac{5}{3} - \frac{1}{2}}$$

Thereby, one obtains

$$\left|I(z,\eta)\right| < C_1 \times C_2 |\vartheta|^{-\frac{n}{3} - \frac{5}{3} - \frac{1}{2}} e^{-2\delta|\vartheta|}$$

and, if Θ_0 is chosen so large that

$$|\vartheta|^{-\frac{n}{3} - \frac{5}{3} - \frac{1}{2}} < e^{\delta|\vartheta|}$$

finally

$$\left|I(z,\eta)\right| < \text{const} \times e^{-\delta|\vartheta|}$$

For $0 < \zeta < -\frac{n}{3} + m + \chi$ and constant $\eta \epsilon \Omega$, the integrals (II) and (IV) thus converge toward zero, over the horizontal sections of the path W for $\Theta \to \infty$.

Thereby, (49) becomes for $\Theta \to \infty$

$$\int_W = \int_{-i\infty}^{+i\infty} - \int_{-\frac{n}{3} + m + \chi - i\infty}^{-\frac{n}{3} + m + \chi + i\infty}$$

or, if the integral values are substituted

$$\tilde{h}_{n,m}(\eta) = h_n(\eta) - \eta \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)} \frac{1}{2\pi i} \int_{-\frac{n}{3}+m+X-i\infty}^{-\frac{n}{3}+m+X+i\infty} \zeta(z,\eta)\, dz \tag{51}$$

In order to show that $\tilde{h}_{n,\infty}(\eta)$ is, for $\eta\epsilon\Omega$, the asymptotic expansion of $h_n(\eta)$ for $\eta \to \infty$, we must prove that for every constant m

$$\lim_{\eta \to \infty} \left[h_n(\eta) - \tilde{h}_{n,m}(\eta)\right] \eta^{-n+3m-2} = 0 \tag{52}$$

is valid. For this, we must estimate - because of (51) - the integral

$$\left[h_n(\eta) - \tilde{h}_{n,m}(\eta)\right] \eta^{-n+3m-2} = \eta^{-n+3m-1} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{n}{3}\right)} \int_{-\frac{n}{3}+m+X-i\infty}^{-\frac{n}{3}+m+X+i\infty} \zeta(z,\eta)\, dz \tag{53}$$

for constant m and $\eta \to \infty$ $(\eta\epsilon\Omega)$. First, if we write for abbreviation

$$M = -\frac{n}{3} + m + X$$

we obtain

$$\left|\frac{1}{2\pi i} \int_{M-i\infty}^{M+i\infty} \zeta(z,\eta)\, dz\right| \leqq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\zeta(M + i\vartheta,\eta)\right| d\vartheta \tag{54}$$

From (50) then follows

$$\left|I(M + i\vartheta,\eta)\right| = H(\eta) \frac{e^{\left(M - \frac{1}{2} - \frac{n+5}{3}\right)\ln|M+i\vartheta|}\, e^{-\vartheta(\arg(M+i\vartheta)-3 \arg \eta)}\, e^{\underline{R}\mu(M+i\vartheta)}}{\left|-3(M + i\vartheta) + 1\right|}$$

with

$$H(\eta) = \sqrt{2\pi}\; e^{-M(1 - \ln 3)}\, |\eta|^{-3M+1}$$

$H(\eta)$ may be put ahead of the integral, the rest may be estimated, in analogy to the former procedure, by

$$\left|I(M + i\vartheta,\eta)\right| \leqq H(\eta)C\left|M + i\vartheta\right|^{M - \frac{1}{2} - \frac{n+5}{3}} e^{-|\vartheta|(|\arg(M+i\vartheta)| \pm 3 \arg \eta)}$$

where C is a constant, and the upper sign applies again for $\vartheta < 0$, lower for $\vartheta > 0$. We now split up the integral (54) into

$$\left|\frac{1}{2\pi i}\int_{M-i\infty}^{M+i\infty} \zeta(z,\eta)\ dz\right| \leqq \frac{C}{2\pi} H(\eta)\left[\int_{-\infty}^{-\Theta_0} \zeta^*(\vartheta,\eta)\ d\vartheta + \int_{-\Theta_0}^{+\Theta_0} \zeta^*(\vartheta,\eta)\ d\vartheta + \int_{\Theta_0}^{\infty} \zeta^*(\vartheta,\eta)\ d\vartheta\right] \tag{55}$$

with

$$I^*(\vartheta,\eta) = \left|M + i\vartheta\right|^{M - \frac{1}{2} - \frac{n+5}{3}} e^{-|\vartheta|(|\arg(M+i\vartheta)| \pm 3 \arg \eta)}$$

Again because of $|\arg \eta| < \frac{\pi}{6} - \epsilon$ and $|\arg (M + i\vartheta)| \to \frac{\pi}{2}$, as above, one must find, for $\vartheta \to \infty$ for given $\delta > 0$, a $\Theta_0 > 0$ in such a manner that

$$0 < 2\delta < |\arg (M + i\vartheta)| \pm 3 \arg \eta$$

is valid for all $|\vartheta| > \Theta_0$ and for each sign. Thereby follows in the same manner

$$I^*(\vartheta,\eta) < Ce^{-\delta|\vartheta|} \quad \text{for} \quad |\vartheta| > \Theta_0, \quad m \quad \text{constant}, \quad \eta\epsilon\Omega$$

that is, the two improper integrals on the right side of (55) are certainly limited. This applies, of course, also for the proper integral and we obtain therefore from (53) altogether

$$\left|h_n(\eta) - \tilde{h}_{n,m}(\eta)\right| \ |\eta|^{-n+3m-2} \leqq \text{const}\ |\eta|^{-n+3m-1} H(\eta) \leqq \text{const}\ |\eta|^{-3\chi}$$

Thereby (52) is actually satisfied for every constant m and $|\eta| \to \infty$ with $\eta\epsilon\Omega$, that is, as asserted before, $\tilde{h}_{n,\infty}(\eta)$ is in Ω the asymptotic expansion of $h_n(\eta)$ for $\eta \to \infty$.

In exactly he same manner one proves that $\tilde{k}_{n,\infty}(\eta)$ likewise represents the asymptotic expansion of $k_n(\eta)$ for $\eta \to \infty$ with $\eta\epsilon\Omega$.

For later, we note that

$$\tilde{k}_{n,\infty}(\eta) = \frac{\Gamma\left(\frac{1}{3}\right)^2}{3^{2/3}\Gamma\left(\frac{2}{3}\right)^2} \tilde{k}_{n,\infty}(\eta) + \frac{4\Gamma\left(\frac{1}{3}\right)}{3^{7/3}\Gamma\left(\frac{2}{3}\right)} \left(\bar{\Psi}\left(\frac{4}{3}\right) - \bar{\Psi}\left(\frac{1}{3}\right)\right)\eta \tag{56}$$

as can be easily checked. This means that there exist solutions of (40) which for $\eta \to \infty$ are asymptotically zero.

4. SOLUTION OF THE INHOMOGENEOUS EQUATIONS FOR THE FIRST COEFFICIENTS $f_n(\tau)$

(a) Function $f_1(\tau)$

With use of the variables $\eta = \sqrt[3]{9a_1}\tau$, there applies, according to (37) and (39), for the coefficient $f_1(\tau) = g_1(\eta)$ the differential equation

$$g_1''' + \eta^2 g_1'' - 3\eta\, g_1' + 3g_1 = -3\,\frac{p_0}{a_1} \tag{57}$$

under the boundary conditions

$$g_1(0) = v_0 \quad (> 0), \quad g_1'(0) = 0, \quad \lim_{\eta \to \infty} \frac{g_1'(\eta)}{\eta^2} = 3\,\frac{a_2}{a_1} \tag{58}$$

A solution of the inhomogeneous equation (57) which satisfies the first two boundary conditions (58) may be found with a polynomial expression of the third order to be

$$v_0 - \frac{a_1 v_0 + p_0}{2a_1}\eta^3$$

Thus, the most general solution of (57) which satisfies those two boundary conditions reads, according to section 3

$$g_1(\eta) = v_0 - \frac{a_1 v_0 + p_0}{2a_1}\eta^3 + \delta_1 h_1(\eta) \tag{59}$$

The constant δ_1 is to be determined by the third boundary condition (58). According to (47a), the asymptotic expansion of $h_1(\eta)$ reads

$$h_1(\eta) \sim \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}}\eta^3 + \frac{1}{3}\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\eta - \frac{2\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} \tag{60}$$

thus for $\eta \to \infty$

$$g_1(\eta) \sim \left[\delta_1 \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} - \frac{a_1 v_0 + p_0}{2a_1}\right]\eta^3 + \delta_1 \frac{1}{3}\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\eta - \left[\delta_1 \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} - v_0\right] \tag{61}$$

is valid. To determine δ_1, we need the asymptotic expansion of $\frac{g_1'(\eta)}{\eta^2}$. According to a well-known theorem,[7] the formally differentiated asymptotic expansion of $g_1(\eta)$ represents the asymptotic expansion of $g_1'(\eta)$ if $g_1'(\eta)$ altogether possesses such an expansion. This is the case, however, for from (59) there results first

$$g_1'(\eta) = -3\,\frac{a_1 v_0 + p_0}{2a_1}\eta^2 + \delta_1 h_1'(\eta)$$

and

$$h_1'(\eta) = \int {}_1F_1\left(-\frac{1}{3}, \frac{5}{3}, -\frac{\eta^3}{3}\right) d\eta + \eta\,{}_1F_1\left(-\frac{1}{3}, \frac{5}{3}, -\frac{\eta^3}{3}\right)$$

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[7] See, for instance, K. Knopp [14] or E. Borel [15].

certainly has an asymptotic expansion, according to section 3. Therefore, it is clear that

$$g_1{}'\ (\eta) \sim 3\left(\delta_1 \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} - \frac{a_1 v_0 + p_0}{2a_1}\right)\eta^2 + \delta_1 \frac{1}{3} \sqrt[3]{3}\ \Gamma\left(\frac{1}{3}\right)$$

and thus

$$\frac{g_1{}'(\eta)}{\eta^2} \sim 3\left(\delta_1 \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} - \frac{a_1 v_0 + p_0}{2a_1}\right) + \frac{1}{\eta^2}\ \delta_1 \frac{1}{3} \sqrt[3]{3}\ \Gamma\left(\frac{1}{3}\right)$$

is valid. If we let η approach infinity $\eta \to \infty$, we obtain, using the third boundary condition (58)

$$3\left(\delta_1 \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} - \frac{a_1 v_0 + p_0}{2a_1}\right) = 3\ \frac{a_2}{a_1}$$

or

$$\delta_1 = \frac{3\sqrt[3]{3}}{2\Gamma\left(\frac{2}{3}\right)}\left(\frac{2a_2 + p_0}{a_1} + v_0\right) = \frac{3\sqrt[3]{3}}{2\Gamma\left(\frac{2}{3}\right)}\ \delta_1{}^* \qquad (62)$$

with

$$\delta_1{}^* = \frac{2a_2 + p_0}{a_1} + v_0$$

For the numerical computation of $g_1(\eta)$, we need merely tabulate the function $h_1(\eta)$. We shall try to attain this, first, by numerical integration of the differential equation

$$h_1{}''' + \eta^2 h_1{}'' - 3\eta h_1{}' + 3h_1 = 0$$

under the initial conditions

$$h_1(0) = h_1'(0) = 0, \quad h_1''(0) = 2$$

However, for this differential equation continuation methods like, for instance, those of Adams or Runge-Kutta are, unfortunately, unstable. The reason lies in the coefficients increasing with η which, moreover, have different signs. In order not to lose too much accuracy, one may therefore work only a short piece with this method, that is, one must use as far as possible the power series (44), thus

$$h_1(\eta) = 2 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\eta^{3m+2}}{(3m - 1)(3m + 1)(3m + 2)3^m m!} \tag{63}$$

We calculated, with the aid of (63) the function

$$\bar{h}_1(\eta) = \frac{1}{2} h_1(\eta)$$

and its first two derivatives in the interval $0 \leqq \eta \leqq 2.5$, with tolerable (calculating) time. For $2.5 \leqq \eta \leqq 3.0$, the method of Adams was used, and at $\eta = 3.0$ there appeared, up to 5 digits after the decimal point, agreement with the values of the expansion (60). The tabulation of $\bar{h}_1'(\eta)$ for $0 \leqq \eta \leqq 6.0$ is given in the appendix.

For later purposes we need, furthermore, the asymptotic expansion of the function $f_1(\tau)$. It results immediately from (61) as

$$f_1(\tau) \sim B_{13}\tau^3 + B_{11}\tau + B_{10} \tag{64}$$

with

$$B_{13} = 9a_2 \qquad B_{11} = \frac{3\Gamma\left(\frac{1}{3}\right)}{2\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)} \sqrt[3]{9a_1}\ \delta_1^* \qquad B_{10} = -\frac{2a_2 + p_0}{a_1}$$

(b) Function $f_2(\tau)$

For the coefficient $g_2(\eta) = f_2(\tau)$ there follows from (37) and (39) the differential equation

$$g_2''' + \eta^2 g_2'' - 4\eta g_2' + 4g_2 = \frac{1}{\sqrt[3]{9a_1}} \left(-3g_1 g_1'' + 2g_1'^2\right) \tag{65}$$

with the boundary conditions

$$g_2(0) = g_2'(0) = 0, \quad \lim_{\eta \to \infty} \frac{g_2'(\eta)}{\eta^3} = \frac{\sqrt[3]{3}\, 3a_3}{a_1^{4/3}} \tag{66}$$

In order to make the functions to be tabulated for the calculation of $g_2(\eta)$ independent of the data of the special problem, we split up the right side of (65) as follows:

$$\frac{1}{\sqrt[3]{9a_1}} \left(-3g_1 g_1'' + 2g_1'^2\right) = \frac{\delta_1^2}{\sqrt[3]{9a_1}} \left(-3h_1 h_1'' + 2h_1'^2\right) +$$

$$\sqrt[3]{3}\, \delta_1 \frac{a_1 v_0 + p_0}{a_1^{4/3}} \left(\frac{h_1''}{2} \eta^3 - 2h_1'\eta^2 + 3h_1\eta\right) -$$

$$\frac{\sqrt[3]{3}\, \delta_1 v_0}{a_1^{1/3}} h_1'' + 3\sqrt[3]{3}\, v_0 \frac{a_1 v_0 + p_0}{a_1^{4/3}} \eta \tag{67}$$

and consider, correspondingly, the differential equations

$$\gamma_1''' + \eta^2 \gamma_1'' - 4\eta\gamma_1' + 4\gamma_1 = -3h_1 h_1'' + 2h_1'^2 \tag{68a}$$

$$\gamma_2''' + \eta^2 \gamma_2'' - 4\eta\gamma_2' + 4\gamma_2 = \frac{h_1''}{2} \eta^3 - 2h_1'\eta^3 + 3h_1\eta \tag{68b}$$

$$\gamma_3''' + \eta^2 \gamma_3'' - 4\eta\gamma_3' + 4\gamma_3 = h_1'' \tag{68c}$$

$$\gamma_4''' + \eta^2 \gamma_4'' - 4\eta\gamma_4' + 4\gamma_4 = \eta \tag{68d}$$

in every case under the initial conditions

$$\gamma_\nu(0) = \gamma_\nu'(0) = \gamma_\nu''(0) = 0 \qquad (\nu = 1,2,3,4) \tag{69}$$

For the equation (68d) there results as the solution of this initial-value problem

$$\gamma_4(\eta) = \frac{\eta^4}{24} \tag{70}$$

Let $\gamma_1 = \gamma_1(\eta)$, $\gamma_2 = \gamma_2(\eta)$, $\gamma_3 = \gamma_3(\eta)$ be the desired solutions of the other equations. Then

$$\bar{g}_2(\eta) = \frac{\delta_1^{\ 2}}{\sqrt[3]{9a_1}} \gamma_1(\eta) + \sqrt[3]{3}\, \delta_1 \frac{a_1 v_0 + p_0}{a_1^{\ 4/3}} \gamma_2(\eta) - \frac{\sqrt[3]{3}\, \delta_1 v_0}{a_1^{\ 1/3}} \gamma_3(\eta) + 3 \sqrt[3]{3} \frac{a_1 v_0 + p_0}{a_1^{\ 4/3}} \frac{\eta^4}{24} \tag{71}$$

is evidently a solution of (65) with $\bar{g}_2(0) = \bar{g}_2'(0) = \bar{g}_2''(0) = 0$, and consequently

$$g_2(\eta) = \bar{g}_2(\eta) + \delta_2 h_2(\eta) \tag{72}$$

represents the most general solution of (65) which satisfies the first two boundary conditions.

In order to determine δ_2, we must again investigate the asymptotic behavior of the solution $g_2(\eta)$. First, there is, according to (47a), (47b) for $\eta \to \infty$

$$h_2(\eta) \sim \tilde{h}_{2,\infty}(\eta) \equiv \frac{\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{6\Gamma\left(\frac{1}{3}\right)}\eta^4 +$$

$$\frac{4\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{9\Gamma\left(\frac{1}{3}\right)}\left[-\ln 3 + \bar{\psi}(2) - \bar{\psi}\left(\frac{1}{3}\right) - \bar{\psi}\left(\frac{4}{3}\right)\right]\eta + \frac{4\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}\eta \ln \eta +$$

$$\frac{4\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}\sum_{n=2}^{\infty}\frac{\left[1.4 \ldots (3n-5)\right]\left[(-1)\,2.5 \ldots (3n-7)\right]}{n!\,3^n(1-n)\eta^{3n-4}} \tag{73}$$

It is therefore necessary to obtain information on the asymptotic behavior of $\gamma_1(\eta)$, $\gamma_2(\eta)$, $\gamma_3(\eta)$. For this we use an approach from Poincaré's theory of asymptotic series.[8] If one replaces in the three differential equations (68a-68c) the right sides every time by their asymptotic expansions, there originate the pertaining so-called asymptotic differential equations

$$\tilde{\gamma}_1''' + \eta^2\tilde{\gamma}_1'' - 4\eta\tilde{\gamma}_1' + 4\tilde{\gamma}_1 = -\frac{2}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\eta^2 + 4\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\right]^2\eta + \frac{2}{9}\left(\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\right)^2 \tag{74a}$$

$$\tilde{\gamma}_2''' + \eta^2\tilde{\gamma}_2'' - 4\eta\tilde{\gamma}_2' + 4\tilde{\gamma}_2 = \frac{1}{3}\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\eta^2 - \frac{2\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\eta \tag{74b}$$

$$\tilde{\gamma}_3''' + \eta^2\tilde{\gamma}_3'' - 4\eta\tilde{\gamma}_3 + 4\tilde{\gamma}_3 = \frac{2\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\eta \tag{74c}$$

One understands now immediately that the asymptotic expansion of a solution of (68)[9] must formally satisfy the corresponding asymptotic

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[8] See, for instance, E. Borel [15].

[9] As far as such an expansion exists at all; however, we shall always make this assumption.

equation (74). For each of these equations one can immediately give a particular integral, namely

$$\tilde{\gamma}_1 = \frac{1}{6}\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\right]^2 \eta^4 + \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\eta^2 + \frac{1}{18}\left(\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\right)^2 \qquad (75a)$$

$$\tilde{\gamma}_2 = -\frac{\Gamma\left(\frac{2}{3}\right)}{12\sqrt[3]{3}}\eta^4 - \frac{1}{6}\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)\eta^2 \qquad (75b)$$

$$\tilde{\gamma}_3 = \frac{\Gamma\left(\frac{2}{3}\right)}{12\sqrt[3]{3}}\eta^4 \qquad (75c)$$

Hence, we obtain the complete solutions of (74) by addition of an arbitrary linear combination of the three solutions $h_2(\eta)$, $k_2(\eta)$, η of the pertaining homogeneous equation. These linear combinations are, because of (56), for $\eta \to \infty$ asymptotically equal $\bar{c}_1\tilde{h}_{2,\infty}(\eta) + \bar{c}_2\eta$ plus a zero expansion

$$\bar{c}_3 e^{-\eta}\left(\bar{\alpha}_0 + \frac{\bar{\alpha}_1}{\eta} + \frac{\bar{\alpha}_2}{\eta^2} + \ldots\right)$$

Thus, we have obtained in

$$\tilde{\gamma}_\nu(\eta) + \bar{c}_{1\nu}\tilde{h}_{2,\infty}(\eta) + \bar{c}_{2\nu}\eta + \bar{c}_{3\nu}e^{-\eta}\sum_{m=0}^{\infty}\frac{\bar{\alpha}_{m\nu}}{\eta^m} \qquad (\nu=1,2,3) \qquad (76)$$

the most general asymptotic expansions which satisfy the equation (74a-74c). Therefore, especially the asymptotic expansions of our solutions γ_1, γ_2, γ_3 of (68/69) also must have this form. Let there be, for instance

$$\gamma_\nu(\eta) \sim \tilde{\gamma}_\nu(\eta) + c_{1\nu}\tilde{h}_{2,\infty}(\eta) + c_{2\nu}\eta + c_{3\nu}e^{-\eta}\sum_{m=0}^{\infty}\frac{\alpha_{m\nu}}{\eta^m} \qquad (\nu=1,2,3) \qquad (77)$$

If we succeed in numerically calculating the appearing constants $c_{1\nu}$, there follows from (75) and (73) immediately

$$\left.\begin{aligned}
\lim_{\eta \to \infty} \frac{\gamma_1'(\eta)}{\eta^3} &= \frac{2}{3}\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\right]^2 + c_{11}\,\frac{2}{3}\,\frac{\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \\
\lim_{\eta \to \infty} \frac{\gamma_2'(\eta)}{\eta^3} &= -\frac{1}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}} + c_{12}\,\frac{2}{3}\,\frac{\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \\
\lim_{\eta \to \infty} \frac{\gamma_3'(\eta)}{\eta^3} &= \frac{\Gamma\left(\frac{2}{3}\right)}{3\sqrt[3]{3}} + c_{13}\,\frac{2}{3}\,\frac{\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}
\end{aligned}\right\} \tag{78}$$

Thus, with consideration of (71), (73), and the outer boundary condition (66), we obtain from (72) a simple qualifying equation for δ_2 the solution of which reads

$$\delta_2 = \frac{1}{\sqrt[3]{9}a_1}\left[\frac{27\Gamma\left(\frac{1}{3}\right)}{2\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}\,\frac{a_3}{a_1} - \frac{27}{4\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}\,\delta_1{}^{*2}c_{11} - \frac{9\sqrt[3]{3}}{2\Gamma\left(\frac{2}{3}\right)}\,\delta_1{}^*\left(v_0 + \frac{p_0}{a_1}\right)c_{12} + \right.$$

$$\left. \frac{9\sqrt[3]{3}}{2\Gamma\left(\frac{2}{3}\right)}\,\delta_1{}^*v_0c_{13} - \frac{9\Gamma\left(\frac{1}{3}\right)}{2\sqrt[3]{3}\,\Gamma\left(\frac{2}{3}\right)}\,\frac{a_2}{a_1}\,\frac{2a_2 + p_0}{a_1}\right] \tag{79}$$

Finally, we have to determine the constants $c_{1\nu}$. For this purpose, we must first calculate the functions $\gamma_\nu(\eta)$ up to sufficiently large η. As for $h_1(\eta)$, here again the numerical instability of the differential equations requires working with the power series, as far as time required for calculation permits. For the functions we tabulated

$$\bar{h}_2'(\eta) = \frac{1}{2}\,h_2'(\eta), \quad \bar{\gamma}_1'(\eta) = \frac{1}{2}\,\gamma_1'(\eta), \quad \bar{\gamma}_2'(\eta) = 5\gamma_2(\eta), \quad \bar{\gamma}_3'(\eta) = \frac{1}{2}\,\gamma_3'(\eta)$$

we used the series in all cases for $0 \leqq \eta \leqq 2.5$ and passed over to Adams' method only from $\eta = 2.5$ onward. For $\overline{h}_2'(\eta)$ there applies, according to (44)

$$\overline{h}_2'(\eta) = \sum_{\nu=0}^{\infty} (-1)^{\nu+1} \frac{1 \cdot 4 \cdot 7 \ \ldots \ (3\nu - 5)}{5 \cdot 8 \cdot 11 \ \ldots \ (3\nu - 1)} \frac{\eta^{3\nu+1}}{(3\nu + 1)\nu! 3^{\nu}} \tag{80}$$

For the $\overline{\gamma}_\nu'(\eta)$ there results from the differential equations

$$\overline{\gamma}_1' = \sum_{\nu=1}^{\infty} E_{3\nu+1}^{(1)} \eta^{3\nu+1}; \quad \overline{\gamma}_2' = \sum_{\nu=2}^{\infty} E_{3\nu+2}^{(2)} \eta^{3\nu+2}; \quad \overline{\gamma}_3' = \sum_{\nu=0}^{\infty} E_{3\nu+2}^{(3)} \eta^{3\nu+2} \tag{81}$$

where the first coefficients E have the values indicated in the following table.

1. Coefficients of the power series of $\overline{\gamma}_1'(\eta)$.-

$3\nu + 1$	$E_{3\nu+1}$	$3\nu + 1$	$E_{3\nu+1}$
1	0	40	+ 29 684 711 -26
4	+ 83 333 333 - 9	43	+ 45 458 070 -28
7	- 67 460 317 -10	46	- 42 030 747 -30
10	+ 48 941 799 -11	49	+ 10 678 059 -30
13	- 29 674 544 -12	52	- 45 994 712 -32
16	+ 15 232 538 -13	55	+ 14 900 549 -33
19	- 67 289 106 -15	58	- 42 190 905 -35
22	+ 25 916 779 -16	61	+ 10 986 648 -36
25	- 87 786 347 -18	64	- 26 944 258 -38
28	+ 26 232 353 -19	67	+ 63 044 703 -40
31	- 68 793 432 -21	70	- 14 189 493 -41
34	+ 15 489 412 -22	73	+ 30 826 575 -43
37	- 27 947 829 -24		

Here, 83 333 333 -9 signifies abbreviately: $83\,333\,333 \cdot 10^{-9}$

2. Coefficients of the power series of $\bar{\gamma}_2'(\eta)$.-

$3\nu + 2$	$E_{3\nu+2}$		$3\nu + 2$	$E_{3\nu+2}$	
2	0		38	+ 10 092 101	-23
5	0		41	- 24 052 323	-25
8	+ 44 642 857	-10	44	+ 53 487 992	-27
11	- 45 093 795	-11	47	- 11 149 234	-28
14	+ 32 044 675	-12	50	+ 21 869 789	-30
17	- 18 329 982	-13	53	- 40 511 235	-32
20	+ 88 663 821	-15	56	+ 71 087 162	-34
23	- 37 287 360	-16	59	- 11 849 619	-35
26	+ 13 891 308	-17	62	+ 18 810 840	-37
29	- 46 484 507	-19	65	- 28 503 156	-39
32	+ 14 123 777	-20	68	+ 41 310 492	-41
35	- 39 306 312	-22	71	- 57 376 598	-43

3. Coefficients of the power series of $\bar{\gamma}_3'(\eta)$.-

$3\nu + 2$	$E_{3\nu+2}$		$3\nu + 2$	$E_{3\nu+2}$	
2	+ 50 000 000	- 8	35	+ 71 304 684	-24
5	+ 25 000 000	- 9	38	- 16 576 145	-25
8	- 94 246 032	-11	41	+ 36 085 121	-27
11	+ 45 093 795	-12	44	- 73 841 439	-29
14	- 20 739 108	-13	47	+ 14 252 289	-30
17	+ 87 285 627	-15	50	- 26 026 919	-32
20	- 33 320 536	-16	53	+ 45 095 094	-34
23	+ 11 558 047	-17	56	- 74 320 380	-36
26	- 36 609 952	-19	59	+ 11 677 902	-37
29	+ 10 648 904	-20	62	- 17 531 552	-39
32	- 28 602 324	-22	65	+ 25 195 265	-41

We shall try to determine the coefficients $c_{1\nu}$ in such a manner that the values of the functions $\gamma_\nu(\eta)$ and their first two derivatives for large η agree with the values calculated from (77). The equations (77) represent together with the equations following from them by single or double differentiation with respect to η, a linear system of equations for $c_{1\nu}$, $c_{2\nu}$, and $c_{3\nu}$, for constant $\eta = \eta_0$ and for every $\nu = 1, 2, 3$. Unfortunately, however, the coefficient of $c_{3\nu}$, namely the zero expansion, is not known numerically. But with increasing η this coefficient decreases more and more so that it may finally be neglected. We therefore consider now the system of equations

$$\gamma_\nu(\eta_0) = \tilde{\gamma}_\nu(\eta_0) + c_{1\nu}\tilde{h}_{2,\infty}(\eta_0) + c_{2\nu}\eta$$

$$\gamma_\nu{}'(\eta_0) = \tilde{\gamma}_\nu{}'(\eta_0) + c_{1\nu}\tilde{h}_{2,\infty}'(\eta_0) + c_{2\nu}$$

$$\gamma_\nu{}''(\eta_0) = \tilde{\gamma}_\nu{}''(\eta_0) + c_{1\nu}\tilde{h}_{2,\infty}''(\eta_0)$$

The $c_{1\nu}$ calculated from the first two equations agrees, for sufficiently larger η_0, better and better with the value found from the last equation. This agreement represents evidently a measure for the damping of the zero expansion in (77). It was attained, up to 5 digits behind the decimal point, at $\eta_0 = 3.0$ for $\bar{\gamma}_3$ and at $\eta_0 = 3.5$ for $\bar{\gamma}_1$, $\bar{\gamma}_2$. The values

$$\left.\begin{aligned} \tfrac{1}{2}\,c_{11} &= -1.12150, \quad 5c_{12} = 8.46731, \quad \tfrac{1}{2}\,c_{13} = 0.47041 \\ \tfrac{1}{2}\,c_{21} &= -0.41497, \quad 5c_{22} = -0.90750, \quad \tfrac{1}{2}\,c_{23} \doteq -0.12001 \end{aligned}\right\} \qquad (82)$$

resulted. The tabulations of the functions $\bar{h}_2{}'(\eta)$, $\bar{\gamma}_1{}'(\eta)$, $\bar{\gamma}_2{}'(\eta)$, $\bar{\gamma}_3{}'(\eta)$ for $0 \leqq \eta \leqq 6.0$ are given in the appendix.

For what follows we need, in addition, the asymptotic expansion of the function $f_2(\tau)$. It is obtained after an easy calculation if the asymptotic expansions (47 a/b) and (77) of $h_2(\eta)$ and $\gamma_\nu(\eta)$ $(\nu = 1, 2, 3)$ are substituted into (73) and we then pass over to the former variable τ. There results

$$f_2(\tau) \sim B_{24}\tau^4 + B_{22}\tau^2 + \bar{B}_{22}\tau \ln \tau + B_{21}\tau + B_{20} + \sum_{m=1}^{\infty} \frac{B_{21} - m}{\tau^m} \qquad (83)$$

with

$$B_{24} = \frac{81}{4} a_3$$

$$B_{22} = \frac{9\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}{2\Gamma\left(\frac{2}{3}\right)} \frac{a_2}{a_1^{2/3}} \left(\frac{2a_2 + p_0}{a_1} + v_0\right)$$

$$\bar{B}_{22} = 18\frac{a_3}{a_1} - 6\frac{a_2\left(2a_2 + p_0\right)}{a_1^2}$$

$$\left.\begin{aligned} B_{21} = &\left(6\frac{a_3}{a_1} - 2\frac{a_2\left(2a_2 + p_0\right)}{a_1^2}\right)\left(\ln a_1 + \ln 3 + \bar{\psi}(2) - \bar{\psi}\left(\frac{1}{3}\right) - \bar{\psi}\left(\frac{4}{3}\right)\right) - \\ &0.41497\,\frac{9}{2}\left[\frac{\sqrt[3]{3}}{\Gamma\left(\frac{2}{3}\right)}\right]^2\left[\frac{2a_2 + p_0}{a_1} + v_0\right]^2 - \\ &0.90750\,\frac{9\sqrt[3]{3}}{10\Gamma\left(\frac{2}{3}\right)}\,\frac{a_1 v_0 + p_0}{a_1}\left(\frac{2a_2 + p_0}{a_1} + v_0\right) + \\ &0.12001\,\frac{9\sqrt[3]{3}}{\Gamma\left(\frac{2}{3}\right)}\,v_0\left(\frac{2a_2 + p_0}{a_1} + v_0\right) \end{aligned}\right\} \quad (83a)$$

$$B_{20} = \frac{1}{8}\left[\frac{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2 \frac{1}{a_1^{1/3}}\left[\frac{2a_2 + p_0}{a_1} + v_0\right]^2$$

(c) Remarks on the Function $f_3(\tau)$

For the coefficient $g_3(\eta) = f_3(\tau)$ there follows from (37) and (39) the differential equation

$$g_3''' + \eta^2 g_3'' - 5\eta g_3' + 5g_3 = \frac{1}{\sqrt[3]{\ }}\left(-4g_2 g_1'' + 5g_2' g_1' - 3g_1 g_2''\right) \quad (84)$$

with the boundary conditions

$$g_2(0) = g_2'(0) = 0 \qquad \lim_{\eta \to \infty} \frac{g_3'(\eta)}{\eta^4} = 3^{5/3} \frac{a_4}{a_1^{5/3}} \tag{85}$$

If one wants to reduce the calculation of $g_3(\eta)$ again to certain universal functions, one has to split up the right side of (84), for instance, in the following manner

$$\frac{1}{\sqrt[3]{9a_1}} \left(-4g_2g_1'' + 5g_2'g_1' - 3g_1g_2''\right) = \frac{1}{(9a_1)^{2/3}} \Big[3\delta_1^2\beta_1K_1(\eta) +$$

$$9\delta_1\beta_1^2K_2(\eta) - 9\delta_1v_0\beta_1K_3(\eta) + 3\sqrt[3]{9a_1}\,\delta_2\beta_1K_4(\eta) - \frac{3}{2}\,\delta_1v_0\beta_1K_5(\eta) +$$

$$\delta_1^3K_6(\eta) + 3\delta_1^2\beta_1K_7(\eta) + 3\delta_1^2v_0K_8(\eta) + 4\sqrt[3]{9a_1}\,\delta_1\delta_2K_9(\eta) -$$

$$3\delta_1^2v_0K_{10}(\eta) - 9\delta_1v_0\beta_1K_{11}(\eta) - 9\delta_1v_0^2K_{12}(\eta) -$$

$$3\sqrt[3]{9a_1}\,v_0\delta_2K_{13}(\eta) + \frac{3}{8}\,v_0^2\beta_1K_{14}(\eta)\Big]$$

where δ_1 and δ_2 denote the expressions (62) and (79),

$$\beta_1 = \frac{a_1v_0 + p_0}{a_1}$$

and, finally, the 14 functions $K_\nu(\eta)$ have the form

$$K_\nu(\eta) = \gamma_\nu''\eta^3 - 5\gamma_\nu'\eta^2 + 8\gamma_\nu\eta \qquad (\nu=1,2,3) \tag{87a}$$

$$K_4(\eta) = h_2''\eta^3 - 5h_2'\eta^2 + 8h_2\eta \tag{87b}$$

$$K_5(\eta) = h_1''\eta^4 - 5h_1'\eta^3 + 9h_1\eta^2 \tag{87c}$$

$$K_\nu(\eta) = -4\gamma_\nu h_1'' + 5\gamma_\nu' h_1' - 3\gamma_\nu'' h_1 \qquad (\nu=6,7,8) \tag{87d}$$

$$K_9(\eta) = -4h_2 h_1'' + 5h_2' h_1' - 3h_2'' h_1 \tag{87e}$$

$$K_\nu(\eta) = \gamma_\nu'' \qquad (\nu=10,11,12) \tag{87f}$$

$$K_{13}(\eta) = h_2'' \tag{87g}$$

$$K_{14}(\eta) = -36\eta^2 \tag{87h}$$

Thereby, we have split up (84) into the 14 universal differential equations

$$\omega_\nu''' + \eta^2\omega_\nu'' - 5\eta\omega_\nu' + 5\omega_\nu = K_\nu(\eta) \qquad (\nu=1,2,\ldots,14) \tag{88}$$

which have first to be considered under the initial conditions

$$\omega_\nu(0) = \omega_\nu'(0) = \omega_\nu''(0) = 0 \tag{89}$$

The number of these differential equations may be slightly reduced: In (86), K_1 and K_7, also K_3, K_5, and K_{11}, and furthermore K_8 and K_{10} have the same coefficient, except for one constant numerical factor; that is, one may combine the corresponding differential equations. We did not do this only for reasons of symmetry.

The last equation (88) has as the solution to the initial conditions (89)

$$\omega_{14} = 12\eta^2$$

Calculation of the remaining 13 functions $\omega_\nu(\eta)$ will be possible only by numerical integration of the differential equations (88) which would probably be somewhat troublesome, again because of the numerical instability of these equations. However, if one then combines the $\omega_\nu(\eta)$ into a function $\bar{g}_3(\eta)$ in the same manner as had been done with the $K_\nu(\eta)$ in (86), $\bar{g}_3(\eta)$ evidently represents an integral of (84) for which

$$\bar{g}_2(0) = \bar{g}_2'(0) = \bar{g}_2''(0) = 0$$

is valid. We have therefore as the most general solution of (84) which satisfies the first two boundary conditions (85)

$$g_3(\eta) = \bar{g}_3(\eta) + \delta_3 h_3(\eta)$$

The still free constant δ_3 is again to be determined from the remaining outer boundary condition. For this purpose, one has to find, exactly as for $g_2(\eta)$, the asymptotic expansions of the universal functions $\omega_\nu(\eta)$. However, we shall not carry this out here.

5. ASYMPTOTIC EXPANSION OF THE STREAM FUNCTION

Our method for solving the boundary-layer problem (13), (14) used so far consisted in expanding the stream function in dependence on the variables

$$\sigma = \sqrt[3]{x - x_0}, \quad \tau = \frac{y}{3\sqrt[3]{x - x_0}}$$

into a power series

$$\psi(x,y) = \sigma^2 f(\sigma,\tau) = \sum_{n=0}^{\infty} f_n(\tau)\sigma^{n+2} \tag{90}$$

For any constant τ, the range of convergence of this series will be an interval $0 \leqq \sigma \leqq \sigma_1(\tau)$. For $\tau \to \infty$, certainly $\sigma_1(\tau) = O\left(\frac{1}{\tau}\right)$ will be valid. On the one hand, the $f_n(\tau)$ tend toward infinity like τ^{n+2}, on the basis of the boundary condition (34c). On the other hand, with the simultaneous limiting process $\sigma \to 0$, $\tau \to \infty$, (90) is transformed with $\sigma\tau = \frac{y}{3}$ = constant into the power series of the entrance profile $\tilde{u}(y)$ which is convergent, according to assumption. Hence, there follows altogether that the series (90), for retained small σ, is usable solely for small τ (that is, therefore, small y). As is frequently necessary in boundary-layer theory, we require for calculation of the stream function for large y an asymptotic expansion of ψ.

In order to obtain a clue for this, we shall investigate the behavior of (90) in the limiting process $\sigma \to 0$, $\tau \to \infty$, $\sigma\tau$ = constant, as far as this limiting process takes place in the range of convergence. Since, therefore, τ, in particular, is to tend toward infinity, $\tau \to \infty$, we may - as is usual in the theory of asymptotic series - substitute for the functions $f_n(\tau)$ in (90) their asymptotic expansions. This gives, according to (64) and (83), the beginning to the asymptotic series

$$\sigma^2 f(\sigma,\tau) \sim \frac{9}{2} a_1 \sigma^2 \tau^2 + \sigma^3 \left(B_{13}\tau^3 + B_{11}\tau + B_{10}\right) + \sigma^4 \left(B_{24}\tau^4 + B_{22}\tau^2 + B_{21}\tau + \bar{B}_{22}\tau \ln \tau + B_{20} + B_{2,-2}\tau^{-2} + \dots\right) + \dots\dots \tag{91}$$

which is valid for the limiting process $\sigma \to 0$, $\tau \to \infty$, $\sigma\tau$ = constant (as long as (90) converges). In order to arrive at the stream function $\psi(x,y)$, one must introduce the x, y or, simpler, the σ, y-coordinate system. This is done by the substitution $\tau = \frac{y}{3\sigma}$ and the following rearrangement according to σ-terms (which is always permissible for asymptotic series)

$$\begin{aligned}\psi \sim &\left(\frac{a_1}{2} y^2 + \frac{B_{13}}{27} y^3 + \frac{B_{24}}{81} y^4 + \dots\right) + \\ &\sigma^2\left(\frac{B_{11}}{3} y + \frac{B_{22}}{9} y^2 + \dots\right) + \sigma^3 \ln \sigma \left(- \frac{\bar{B}_{22}}{3} y + \dots\right) + \\ &\sigma^3\left(B_{10} + \frac{\bar{B}_{22}}{3} y \ln y + \frac{1}{3}\left(B_{21} - \bar{B}_{22} \ln 3\right) y + \dots\right) + \\ &\sigma^4\left(B_{20} + \dots\right) + \dots\dots\end{aligned} \tag{92}$$

Thereby, the limiting process $\sigma \to 0$, $\tau \to \infty$, $\sigma\tau$ = constant has become the simple limiting process $\sigma \to 0$ for constant y. We have already made use of this, inversely, in the derivation of the boundary condition (34c). (92) therefore represents an asymptotic expansion of $\psi(\sigma,y)$ for $\sigma \to 0$ for constant (larger) y [10] of the generalized form

$$\psi \sim \sum_{k=0}^{\infty} \sum_{l=0}^{k} S_{kl}(y) \sigma^k \ln^{k-l} \sigma \tag{93}$$

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[10] For small y, the series (92), corresponding to its derivation, is of course no longer valid, that is, it does not satisfy the two inner boundary conditions (12a), either.

As will be shown in the appendix to this report, it is permissible to use such series in calculating exactly like ordinary asymptotic expansions, if one only uses for them the definition

$$\lim_{\sigma \to 0} \left[\psi - \left(s_\infty + s_{10}\sigma \ln \sigma + \dots s_{MN}\sigma^M \ln^{M-N}\sigma\right)\right] \frac{1}{\sigma^M \ln^{M-N}\sigma} = 0 \qquad (94)$$

- with arbitrary M and $N \leqq M$, and with constant y.

We may now drop the assumption, contained in the derivation of (92) that σ and y may vary only within the range of convergence of (90). Rather, the expansion (92) must represent everywhere where it has meaning also the asymptotic series of the stream function $\psi(\sigma,y)$, since $\psi(\sigma,y)$ can possess only one asymptotic series of the form (93) for $\sigma \to 0$, as shown in the appendix. However, we do not know in what range (92) is defined altogether, since for the time being we cannot make a statement on the convergence of the coefficient expansions. But even if we presuppose this convergence for all y, the stream function for y of arbitrary magnitude can still not be actually calculated from (92): For practical purposes, we always know only a finite number of terms of the coefficient series of (92), since we can of course determine only a finite number of functions $f_n(\tau)$ in (90). It is not even to be expected that one will get far beyond the three functions $f_0(\tau)$, $f_1(\tau)$, $f_2(\tau)$.

In order to be able to use (92) in the calculation of $\psi(\sigma,y)$ for all large y, we must find another type of rule for calculation, which is valid for all y, for the coefficients of the σ terms, instead of the series used until now. This one can actually achieve if one takes into consideration that (92) must satisfy the boundary-layer equations (13). If we write, therefore, (92) abbreviately in the form

$$\psi \sim s_0(y) + s_2(y)\sigma^2 + \bar{s}_3(y)\sigma^3 \ln \sigma + s_3(y)\sigma^3 + s_4(y)\sigma^4 + \dots \qquad (95)$$

and substitute this into (13), we obtain, by comparison of the coefficients of the first four σ expressions occurring, the differential equations[11]

$$s_0's_2' - s_0''s_2 = 0 \qquad (96a)$$

$$s_0'\bar{s}_3' - s_0''\bar{s}_3 = 0 \qquad (96b)$$

[11] This, in turn, corresponds to an idea of S. Goldstein which he used for the calculation of the wake behind the plane plate [12] and, in [16], for the flow in the proximity of the separation point.

$$s_0'\left(s_3' + \frac{1}{3}\bar{s}_3'\right) - s_0''\left(s_3 + \frac{1}{3}\bar{s}_3\right) = p_0 + s_0''' \tag{96c}$$

$$s_0's_4' - s_0''s_4 = \frac{1}{2}\left(s_2s_2'' - s_2'^2\right) \tag{96d}$$

From these, we have to calculate those solutions $s_\nu(y)$ which, for small y, agree with the coefficients of (92). On the basis of the well-known uniqueness theorems for ordinary differential equations, they will then be everywhere equal to these coefficients. With the functions $s_\nu(y)$ found in this manner, (95) now represents the beginning of the desired asymptotic series of $\psi(\sigma,y)$ for $\sigma \to 0$, since, as mentioned above, every function can possess at most one asymptotic expansion of the form (93).

Before carrying out the integration of (96), we should like to make a remark: Entering into the boundary-layer equations a priori an expression (93) for the desired asymptotic expansion of $\psi(\sigma,y)$ is suggested. According to the definition (94), one must then perform, for any two values M and N, the comparison of coefficients in the σ expressions up to $\sigma^M \ln^{M-N}\sigma$. Thereby, one obtains a system of $\binom{M+1}{2} + N + 1$ coupled ordinary differential equations for the coefficients $s_{\infty}, \ . \ . \ . \ , s_{MN}$. However, from this we can never draw a conclusion as to the asymptotic expansion (92) or (95), without knowledge of $\psi(\sigma,y)$, thus of (90), since we do not even know what s_{kl} appear at all and which disappear identically.

Now the differential equations (96) are to be integrated under the initial conditions given by (92). Since, of course, according to our statement of the problem, the stream function ψ for $\sigma \to 0$ must correspond to the entrance profile $\tilde{u}(y)$, we have, first

$$s_0'(y) = \tilde{u}(y) = \sum_{n=1}^{\infty} a_n y^n \tag{97}$$

which, self-evidently, agrees with (92). Thus, there results from (96a)

$$s_2(y) = \lambda_2 s_0'(y) = \lambda_2 \tilde{u}(y) \tag{98}$$

where, because of (92)

$$\lambda_2 = \frac{B_{11}}{3a_1}$$

In exactly the same manner we obtain

$$\bar{s}_3(y) = \bar{\lambda}_3 s_0'(y) = \bar{\lambda}_3 \tilde{u}(y) \tag{99}$$

with

$$\bar{\lambda}_3 = -\frac{\bar{B}_{22}}{3a_1}$$

The differential equation (96c) is - under the assumption $s_0'(y) = \tilde{u}(y) \neq 0$ certainly valid for $y \neq 0$ - equivalent with

$$\frac{s_0'\left(s_3' + \frac{1}{3}\bar{s}_3'\right) - s_0''\left(s_3 + \frac{1}{3}\bar{s}_3\right)}{s_0'^2} = \frac{p_0 + s_0'''}{s_0'^2}$$

whence follows immediately

$$s_3 + \frac{1}{3}\bar{s}_3 = s_0' \int_0^y \frac{p_0 + s_0'''}{s_0'^2}\, dy = \tilde{u}(y) \int_0^y \frac{p_0 + \tilde{u}''}{\tilde{u}^2}\, dy \tag{100}$$

Since the general solution of the homogeneous equation (96c) as well as $\bar{s}_3$ is, according to (99), a multiple of $s_0' = \tilde{u}$, we obtain, after we have put for abbreviation

$$I(y) = \int_0^y \frac{p_0 + \tilde{u}''}{\tilde{u}^2}\, dy$$

altogether

$$s_3(y) = \lambda_3 \tilde{u}(y) + \tilde{u}(y) I(y) \tag{101}$$

As may be easily checked, the beginning of the expansion of $s_3(y)$ reads

$$s_3(y) = -\frac{2a_2 + p_0}{a_1} + \left(\lambda_3 a_1 - \frac{a_2(2a_2 + p_0)}{a_1^2}\right) y + \left(\frac{6a_3}{a_1} - \frac{2a_2(2a_2 + p_0)}{a_1^2}\right) y \ln y +$$
$$\left(\lambda_3 a_2 + \frac{12a_4}{a_1} - \frac{12a_2a_3}{a_1^2} + 3\,\frac{a_2^2 - a_1a_3}{a_1^3}\,(2a_2 + p_0)\right) y^2 + \dots\dots$$

so that

$$\lambda_3 = \frac{a_2\left(2a_2 + p_0\right)}{a_1^3} + \frac{1}{3a_1}\left(B_{21} - \overline{B}_{22} \ln 3\right)$$

Because of (98), the differential equation (96d) may be written in the form

$$s_0{}'s_4{}' - s_0{}'' \, s_4 = \frac{B_{11}{}^2}{18a_1{}^2}\left(s_0{}'s_0{}''' - s_0{}''^2\right)$$

A particular integral is

$$s_4 = \frac{B_{11}{}^2}{18a_1{}^2} s_0{}''$$

therefore we have as a general solution

$$s_4(y) = \frac{B_{11}{}^2}{18a_1{}^2} \tilde{u}' + \lambda_4 \tilde{u} \qquad (103)$$

The constant λ_4 can no longer be determined immediately from (92), unless further coefficients $F_n(\tau)$ in (90) are calculated.

Thereby it is now possible to calculate without further difficulties the desired velocity distribution $u(x,y)$ in a small interval $x_0 \leqq x \leqq x_1$ for all $y \geqq 0$. For small y, this is done directly with the aid of the series (90); for larger y, one has to use the asymptotic expansion (95) resulting from it.

In the derivation of this solution $u(x,y)$ of the boundary-layer equations (13), we did not consider at all the outer boundary condition

$$u(x,\infty) = u_\infty(x) \qquad (104)$$

As mentioned at the start, however, the author has shown in another report [9] that, under certain assumptions, this boundary condition is always automatically satisfied in an interval $x_0 \leqq x \leqq x_1$, if only the entrance profile $\tilde{u}(y)$ correctly adjoins the external flow:

$$u(x_0,\infty) = \tilde{u}(\infty) = u_\infty(x_0) \qquad (105)$$

We shall here not further concern ourselves with the exact presupposition of this theorem but shall simply verify that our solution $u(x,y)$ - as far as we calculated it in the x-direction - for $y \to \infty$ actually satisfies the outer boundary condition (104).

For large y and small σ, the equation (95) is valid for $u(x,y)$, thus with consideration of the formulas found for the $s_\nu(y)$

$$u = \psi_y \sim \tilde{u} + \lambda_2 \tilde{u}'\sigma^2 + \bar{\lambda}_3 \tilde{u}'\sigma^3 \ln \sigma + s_3'(y)\sigma^3 + \left(\lambda_4 \tilde{u}' + \frac{B_{11}{}^2}{18a_1{}^2}\tilde{u}''\right)\sigma^4 + \ldots\ldots \tag{106}$$

For $\tilde{u}(y)$ let us assume first, that (105) is correct. Furthermore, we assume that $\tilde{u}(y)$ with all its derivatives for $y \to \infty$ possesses asymptotic expansions.[12] If for instance

$$\tilde{u}(y) \sim u_\infty(x_0) + \frac{\alpha_1}{y} + \frac{\alpha_2}{y} + \ldots\ldots$$

one obtains from it, as is well known, the asymptotic expansions of the derivatives by formal differentiation with respect to y. Thereby the existence of the boundary values

$$\lim_{y \to \infty} \tilde{u}^{(n)}(y) = 0 \qquad (n=1,2,\ldots)$$

also is guaranteed, and we obtain from (106) for constant σ

$$\lim_{y \to \infty} u(x,y) \sim \lim_{y \to \infty} \tilde{u}(y) + \sigma^3 \lim s_3'(y) + \ldots\ldots$$

$$= u_\infty(x_0) + (x - x_0) \lim_{y \to \infty} s_3'(y) + \ldots\ldots \tag{107}$$

[12]This assumption is fundamental also for the general theorem. It corresponds to the character of the boundary-layer flow which has to make asymptotically the transition to the outer flow.

For the boundary value remaining on the right side, one finds from (96c)

$$u_\infty(x_0) \lim_{y\to\infty} s_3'(y) = p_0 + \lim_{y\to\infty} s_0''\left(s_3 + \frac{1}{3}\bar{s}_3\right)$$

$$= p_0 + \lim_{y\to\infty} s_0'' s_0' I(y)$$

$$= p_0 + u_\infty(x_0) \lim_{y\to\infty} \tilde{u}' I(y)$$

The integral $I(y)$ has, on the basis of our assumptions regarding $\tilde{u}$, for $y \to \infty$ certainly an asymptotic expansion. The latter starts, as may be easily checked, with

$$I(y) \sim \frac{p_0}{u_\infty^2(x_0)} y - \frac{2\alpha_1 p_0}{u_\infty^3(x_0)} \ln y + \text{powers of } \frac{1}{y}$$

Since the expansion of $\tilde{u}'(y)$ reads

$$\tilde{u}'(y) \sim -\frac{\alpha_1}{y^2} - \frac{2\alpha_2}{y^3} - \ldots..$$

one recognizes immediately that

$$\lim_{y\to\infty} \tilde{u}' I(y) = 0$$

and accordingly

$$\lim_{y\to\infty} s_3'(y) = \frac{p_0}{u_\infty(x_0)} = \frac{u_\infty(x_0) u_\infty'(x_0)}{u_\infty(x_0)} = u_\infty'(x_0)$$

(107) becomes therefore actually

$$u(x,\infty) \sim u_\infty(x_0) + u_\infty'(x_0)(x - x_0) + \ldots = u_\infty(x)$$

and, on the basis of the convergence of the series of $u_\infty(x)$, the asymptotic sign may be replaced by an equality sign.

Thereby we have proved the validity of the outer boundary condition (104) for the broken-off series (90) and (95) as far as they approximate the solution $u(x,y)$. If one includes further terms of the series, the proof for them shows perfect analogy.

6. COMPILATION OF THE METHOD FOR PRACTICAL APPLICATION

Our method developed so far represents a solution of the problem formulated in section 1: to calculate the boundary-layer flow in the proximity of a jumplike suction start. However, as also has been said in section 1, the possibilities of application of the method, are thereby, by no means exhausted. As may be confirmed immediately, we have in the derivation of the method never made use of the assumption that the entrance profile $\tilde{u}(y)$ at the point $x = x_0$ of the suction start is a velocity profile of the boundary layer without suction. Accordingly, $\tilde{u}(y)$ may obviously be also a profile of a boundary layer with suction; merely a sudden change of the suction velocity then occurs at $x = x_0$, or the suction ceases abruptly. In any case, our method for calculation of the flow may be used a short distance behind the point of discontinuity $x = x_0$. In order to facilitate the practical solution of these problems, we shall, below, once more compile all necessary formulas.

For the example to be considered, first let the outer velocity distribution be given in the form of a power series

$$u_\infty(x) = \sum_{n=0}^{\infty} u_n(x - x_0)^n \tag{108}$$

of which, it is true, we require only the first coefficients u_0 and u_1. At the point $x = x_0$ the suction velocity $v_0(x)$ is now to become discontinuous; let for instance

$$v(x_0,0) = v_0 \neq \lim_{x \to x_0 - 0} v(x,0) = \tilde{v}_0 \tag{109}$$

be valid. For the special case of sudden suction start - formerly the only permissible one - for instance, $v_0 \neq 0$, $\tilde{v}_0 = 0$; at the sudden end of the suction, in contrast, we have $v_0 = 0$, $\tilde{v}_0 \neq 0$. We shall assume that the calculation of the boundary layer up to the point of discontinuity $x = x_0$ has already been carried out. Thereby, one then knows, in principle, the entrance profile $\tilde{u}(y)$ - unfortunately, however, in general only numerically at the equidistant points $y_n = nl$ $(n = 0,1,2, \ldots)$ of a fixed step interval l. However, we need of $\tilde{u}(y)$, too, at least the beginning of the power series.

$$\tilde{u}(y) = a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + \ldots . \tag{110}$$

In the determination of the occurring coefficients a_n one must always observe the well-known wall restrictions.

$$a_2 = -2u_0u_1 + 2\tilde{v}_0 a_1 \qquad a_3 = 6\tilde{v}_0 a_2 \tag{111}$$

By them, a_2 and a_3 are unequivocally coupled to a_1 or - for $\tilde{v}_0 = 0$ - fully determined; a_1 and a_4, in contrast, remain completely undetermined. a_1 is ascertained, first only approximately, for instance by numerical or graphical differentiation of $\tilde{u}(y)$ in $y = 0$. Hence, a_2 and a_3 may be calculated. For a sufficiently small ordinate $y = y_1$, we then find a first correction of a_1 from the deviations of the first three terms of (110), compared to the value of $\tilde{u}(y_1)$. Then, the coefficient a_4 in first approximation is determined for a somewhat larger $y = y_2$. Finally, with its aid, a second correction of a_1 (thus also of a_2 and a_3) is carried out. In general, thereafter, sufficient accuracy is attained for a_1 whereas the most inaccurate coefficient a_4 is only of small influence, anyhow.[13]

After this preliminary work, the constants which return again and again

$$p_0 = u_0u_1 \qquad \sqrt[3]{9a_1} \qquad \delta_1{}^* = \frac{2a_2 + p_0}{a_1} + v_0$$

$$\beta_1 = v_0 + \frac{p_0}{a_1} \qquad \beta_2 = \frac{2a_2 + p_0}{a_1{}^2}$$

may now be calculated, and with them the coefficients

$$\left.\begin{aligned} A_0 &= \frac{9a_1}{\sqrt[3]{9a_1}} \\ A_{11} &= 1.5\,\sqrt[3]{9a_1}\,\beta_1 \\ A_{12} &= 3.195253\,\sqrt[3]{9a_1}\,\delta_1{}^* \end{aligned}\right\} \tag{112a}$$

---

[13]A detailed representation of the method for determination of the a_n here described, for the special case $\tilde{v}_0 = 0$, may be found in H. Görtler [17].

as well as

$$\left.\begin{aligned} A_{21} &= 5.104819\left(\delta_1^*\right)^2 \qquad A_{22} = 0.958576\ \beta_1\delta_1^* \\ A_{23} &= 9.585757\ v_0\delta_1^* \qquad A_{24} = 1.5\ v_0\beta_1 \\ A_{25} &= 37.03647\ \frac{a_3}{a_1} + 11.45011\left(\delta_1^*\right)^2 - 16.23311\ \beta_1\delta_1^* + \\ &\quad 9.01847\ v_0\delta_1^* - 12.34549\ a_2\beta_2 \end{aligned}\right\} \quad . \tag{112b}$$

Hence we then obtain, with use of

$$\sqrt[3]{9a_1}\ g_1'(\eta) = -A_{11}\eta^2 + A_{12}\bar{h}_1'(\eta) \tag{113a}$$

and

$$\sqrt[3]{9a_1}\ g_2'(\eta) = A_{21}\bar{\gamma}_1' + A_{22}\bar{\gamma}_2' - A_{23}\bar{\gamma}_3' + A_{24}\eta^3 + A_{25}\bar{h}_2' \tag{113b}$$

given in the appendix, thus far a small

$$u(\sigma,\eta) = A_0\eta\ \frac{\sigma}{3} + \sqrt[3]{9a_1}\ g_1'(\eta)\ \frac{\sigma^2}{3} + \sqrt[3]{9a_1}\ g_2'(\eta)\ \frac{\sigma^3}{3} + \ldots\ldots \tag{114}$$

The choice of σ depends on the requirements of the individual examples. In our practical test, we have admitted, at most, numerical values $\sigma \leqq 0.15$, in order to lose as little accuracy as possible. This - fundamentally small - step interval in the x-direction, however, brought us in every case so far beyond the influence of the flow discontinuity at $x = x_0$, that a difference method became again applicable.

If one now introduces into (114) the variable

$$y = \frac{3\sigma}{\sqrt[3]{9a_1}}\ \eta$$

which can be done directly with simple linear interpolation - the desired velocity profile $u(x,y)$ for small y is obtained. How far this $u(x,y)$ must be calculated in the y-direction depends on the individual case. Generally, one will use for (114) the tables of the universal functions up to $\eta = 4.0$ or 5.0. Frequently, however, the automatic junction to the asymptotic expansion of $u(x,y)$ is attained earlier. For y-values beyond this junction the deterioration of the approximation formula (114) becomes slowly noticeable.

For large y, the asymptotic expansion of $u(x,y)$ is now to be set up according to section 5. For this purpose, we calculate first the constants

$$B_{11} = 2.057582 \sqrt[3]{9a_1}\, \delta_1^*$$

$$\bar{B}_{22} = 18 \frac{a_3}{a_1} - 6a_2\beta_2$$

$$B_{21} = (0.333333 \ln a_1 + 1.595155)\bar{B}_{22} - 2.11835 \left(\delta_1^*\right)^2 - 0.86991\, \beta_1\delta_1^* + 1.15039\, v_0\delta_1^* \quad (115)$$

[14]

$$\beta_3 = 12 \frac{a_4}{a_1} - \frac{2}{3}\frac{a_2}{a_1}\bar{B}_{22} - \frac{a_2^2 + 2a_1a_3}{a_1^2}\beta_2$$

---

[14] This contains, by the way, the determination of the values $\bar{\psi}\left(\frac{1}{3}\right)$, $\bar{\psi}\left(\frac{2}{3}\right)$ of the logarithmical derivatives of the gamma function; these values are rarely to be found in tabular compilations. From the well-known formulas

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad \Gamma\left(\frac{x}{3}\right)\Gamma\left(\frac{x+1}{3}\right)\Gamma\left(\frac{x+2}{3}\right) = \frac{2\pi\Gamma(x)}{3^{x-1/2}}$$

one obtains, by logarithmical differentiation and substitution of $x = \frac{1}{3}$ or $x = 1$, a linear system of equations for the two unknowns $\bar{\psi}\left(\frac{1}{3}\right)$, $\bar{\psi}\left(\frac{2}{3}\right)$ the solution of which reads

$$\bar{\psi}\left(\frac{1}{3}\right) = -C - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 \qquad \bar{\psi}\left(\frac{2}{3}\right) = -C - \frac{3}{2}\ln 3$$

($C = 0.577\,215\,66$ Euler's constant.) Correspondingly, we obtain for the expression occurring in (73) and (83)

$$-\ln 3 + \bar{\psi}(2) - \bar{\psi}\left(\frac{1}{3}\right) - \bar{\psi}\left(\frac{4}{3}\right) = -2 + 2\ln 3 + C + \frac{\pi}{\sqrt{3}}$$

$$= 2.588\,239\,61$$

Thereby the first coefficient functions of that expansion read

$$\left.\begin{aligned}
s_0{}' &= \tilde{u}(y) \\
s_2{}' &= \frac{B_{11}}{3a_1}\,\tilde{u}'(y) \\
\bar{s}_3{}' &= -\frac{\bar{B}_{22}}{3a_1}\,\tilde{u}'(y) \\
s_3{}' &= \lambda_3\tilde{u}' + \frac{\tilde{u}'' + p_0}{\tilde{u}} + \tilde{u}\int_0^y \frac{\tilde{u}'' + p_0}{\tilde{u}^2}\,dy
\end{aligned}\right\} \qquad (116)$$

with

$$\lambda_3 = \frac{1}{a_1}\left(\beta_2 + 0.333333\ B_{21} - 0.366204\ \bar{B}_{22}\right)$$

For evaluation of the integral appearing in $s_3{}'$, it is best to split up the latter at an arbitrary but small displacement $\hat{y}$

$$\int_0^y \frac{\tilde{u}'' + p_0}{\tilde{u}^2}\,dy = I(\hat{y}) + \int_{\hat{y}}^y \frac{\tilde{u}'' + p_0}{\tilde{u}^2}\,dy$$

where then

$$I(\hat{y}) = -\beta_2\frac{1}{\hat{y}} + 0.333333\ \bar{B}_{22}\ \ln\hat{y} + \beta_3\hat{y} + \ldots\ldots$$

and for the remaining integral now one of the known formulas of quadrature, as for instance the trapezoidal rule, may be used without difficulty.

If the entrance profile $\tilde{u}$ is given only numerically at the points $y_n = nl$ $(n = 0,1,2, \ldots)$, the calculation of the derivatives $\tilde{u}'$ and $\tilde{u}''$ will possibly cause certain difficulties. According to our experiences, however, it is generally perfectly sufficient to approximate $\tilde{u}'$ and $\tilde{u}''$ by so-called alternating differences of the first or second order. If one writes abbreviately

$$\tilde{u}(y_n) = \tilde{u}_n \qquad (n=0,1,\ldots)$$

$$\nabla_n = \tilde{u}_{n+1} - \tilde{u}_{n-1} \qquad (n=1,2,\ldots)$$

$$\nabla_n{}^2 = \nabla_{n+1} - \nabla_{n-1} \qquad (n=2,3,\ldots)$$

therefore

$$\tilde{u}' = \frac{\nabla_n}{2l} \qquad \tilde{u}'' = \frac{\nabla_n^{\,2}}{4l^2}$$

and the equations (115) take the finite form

$$s_0'\ (y_n) = \tilde{u}_n$$

$$s_2'\ (y_n) = \frac{B_{11}}{6a_1 l}\nabla_n$$

$$\bar{s}_3'(y_n) = -\frac{\bar{B}_{22}}{6a_1 l}\nabla_n$$

$$s_3'(y_n) = L_n + \nabla_n\left[\frac{\lambda_3 + I(y_k)}{2l} + \frac{1}{4}L_k^* + \frac{1}{2}L_{k+1}^* + \ldots + \frac{1}{2}L_{n-1}^* + \frac{1}{4}L_n^*\right]$$

with

$$y_k = \hat{y} \qquad L_n = \frac{1}{\tilde{u}_n}\left(\frac{\nabla_n^{\,2}}{4l^2} + p_0\right) \qquad L_n^* = \frac{L_n}{\tilde{u}_n}$$

The formula for $s_3'(y)$ is valid only for $n \geqq 2$. For smaller values of y - as far as such values are needed altogether - one uses best the power-series expansion (102).

With the aid of the $s_n'(y)$ we now obtain immediately

$$u(\sigma,y) = s_0'(y) + s_2'(y)\sigma^2 + s_3(y)\sigma^3 \ln \sigma + s_3'(y)\sigma^3 + \ldots.. \tag{118}$$

There σ is to be selected exactly as for (114). These $u(x,y)$ always adjoin automatically and correctly the values from (114) so that we now have the velocities u for the entire range of the y.

7. EXAMPLES

1st example: Boundary layer on a plane plate in a longitudinal flow with constant outer velocity distribution u_∞ for jumplike suction start at a point $x = x_0$.

We use for the introduction of the dimensionless coordinates, according to (12), $L = x_0$ and $U = u_\infty$ as characteristic quantities. Then the outer velocity distribution u_∞^* [15] always has the value 1 and the suction starts at the point $x'' = 1$. Up to this point, the well-known Blasius plate boundary layer could freely develop; tabulation of its values may be found for instance in H. Schlichting [1] on p. 103. We obtain accordingly for the inlet profile $\tilde{u}^*(y^*)$ is $x^* = 1$, as may be immediately confirmed

$$\tilde{u}^*(y^*) = 0.33206\ y^* - 0.00230\ y^{*4} + - \ldots..$$

As suction law, we selected successively

$$v_0^*(x^*) = \frac{v_0(x)}{u_\infty}\sqrt{\frac{u_\infty x_0}{\nu}} = 0.5,\ 1.0,\ 1.5$$

In each of these three cases we now used our calculation method for the transition over the flow discontinuity at $x^* = 1$, up to the point $x^* = 1.0015$. Here the influence of the discontinuity had diminished so far that for the further calculation the difference method described in appendix 2 could be applied. A criterion for this applicability is the form of the second derivative of the velocity profile with respect to y^*. Corresponding to the wall restrictions (111) $u^*_{y^*y^*}$ on $x^* = 1$ becomes discontinuous. This discontinuity is smoothed out for instance in the manner shown in figure 1 for our example of the suction value $v_0^* = 0.5$. The difference method works satisfactorily only when the minimum of $-u^*_{y^*y^*}$ lies at least at the third or fourth grid point, without the step interval l in the y*-direction being too small.

For each of the three suction values the calculation thus began at $x^* = 1.0015$ with the difference method. We selected as the step interval in the y*-direction first $l = 0.2$ and later $l = 0.4$, whereas in the x*-direction - corresponding to the slow diminishing of the influence of the discontinuity - h increased from 0.0035 (at the first step) to 0.1.

For the suction value $v_0^* = 0.5$, we performed this calculation up to the point $x^* = 1.15$ where - as figure 1 shows - the second derivative $u^*_{y^*y^*}$ had been approximately smoothed out. Figure 2 shows the velocity curves obtained in this manner for constant distance from the

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[15] The dimensionless coordinates introduced according to (12) will be provided with an asterisk, corresponding to their definition, since the simplified notation without asterisk used so far now could give rise to confusion with the original variables which were dimensional quantities.

wall. One recognizes from it how the discontinuity of the v-boundary values at $x^* = 1$, $y^* = 0$ has been propagated into the flow along the characteristic line $x^* = 1$.

Exactly as for $v_0^* = 0.5$, we calculated also for $v_0^* = 1.5$ up to the point $x^* = 1.15$. For $v_0^* = 1.0$, in contrast, we used the difference method up to the time where the velocity profiles within the scope of calculating accuracy (about 1 percent) had become equal to the asymptotic suction profile

$$u_a^*(y^*) = 1 - e^{-v_0^* y^*}$$

This occurred at $x^* = 2.3$. Figure 3 shows some of the velocity profiles obtained and figure 4 the pertaining streamline pattern. In figure 4, too, one recognizes again the discontinuity along the characteristic line $x^* = 1$.

The four following figures serve to illustrate the differences for the various suction values.

For the suction value $v_0^* = 1.5$ we assumed furthermore that behind $x^* = 1.15$ the wall is again impermeable and that, therefore, the suction ends suddenly at this point. For surmounting the flow discontinuity originating thereby at $x^* = 1.15$ we applied again our method developed during this investigation. For the last suction profile at $x^* = 1.15$, the beginning of the power series

$$u^*(1.15\ y^*) = 1.0430\ y^* - 0.7823\ y^{*2} + 0.3911\ y^{*3} - 0.0035\ y^{*4} + \ldots..$$

determined according to section 6, was used. For further calculation from the point $x^* = 1.152$ onward Görtler's difference method was used again, this time in the form improved by H. Witting, with the step intervals $l = 0.4$ and h between 0.003 and 0.1. Figures 9 to 11 show a few results of these calculations.

From figure 11 one recognizes that the displacement thickness is subjected to a long-lasting reduction by the suction slot whereas for the wall shear stress the effect of the suction rapidly ends again.

2nd example: Boundary layer on the circular cylinder with pressure distribution according to potential theory for sudden suction start at a point $x = x_0$.

Let R be the radius of the circular cylinder and U_0 the velocity of the approach flow. When the dimensionless coordinates (12) are referred to the quantities $L = \frac{R}{2}$ and $U = U_0$, the velocity distribution at the wall according to potential theory is

$$u_\infty^*(x^*) = 2 \sin \frac{x^*}{2}$$

As the point of the suction start we selected $x_0^* = 3.0$ which corresponds to a center angle $\varphi = 87.38^\circ$. Up to this value $x^* = 3.0$, the boundary layer could be calculated without difficulty, with the aid of the Blasius series. The tabulations of A. Ulrich [18] were used.

For the suction, starting jumplike at $x^* = 3.0$, we assumed first $v_0^* = 0.5$. For surmounting the flow discontinuity we had to use our method developed during this study, in the region $3.0 \leqq x \leqq 3.002$. Behind $x^* = 3.002$ the difference method proved again workable.

In order to shift the separation point as far rearward as possible, it was suitable also to continuously increase, with increasing x^*, the suction value v_0^*. A difference method is perfectly capable of dealing with such a continuous change of the v-boundary value $v_0^*(x^*)$ as long as the variations $v_0^*(x^* + h) - v_0^*(x^*)$ originating at each step of the process assume at most numerical values up to 0.1, without a smaller-than-usual step interval h being chosen. However, if these variations exceed 0.1 or if, in order to prevent that, h would have to be very small, the difference method fails rather rapidly. Then one is solely dependent on a jumplike increase of v_0^*, with use of our calculation method for surmounting the discontinuities.

Applying what has been said we now calculated the boundary layer, beginning at $x^* = 3.002$, with the aid of the difference method, for the following suction law:

$$v_0^* = \begin{cases} 0.5 + (x^* - 3.0)\,0.75 & \text{for } 3.0 \leqq x^* \leqq 3.1 \\ 1.1 + (x^* - 3.8) & \text{for } 3.8 \leqq x^* \end{cases}$$

The step interval in the y*-direction was $l = 0.4$, whereas h varied at first, in the range $3.0 \leqq x^* \leqq 3.1$, between 0.003 and 0.02, and, for $x^* \geqq 3.1$, between 0.05 and 0.1.

For this calculation, separation occurred at the point $x^* = 5.1$. This corresponds to a center angle $\varphi = 146.1^\circ$. For a circular cylinder without suction, in contrast, H. Witting [19] found separation as early as at $x^* = 3.80$, that is, $\varphi = 108.9^\circ$.

Figures 12 to 15 serve to illustrate the results.

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APPENDIX I

TABLES OF THE FUNCTIONS $\bar{h}_1'$, $\bar{h}_2'$, $\bar{\gamma}_1'$, $\bar{\gamma}_2'$, $\bar{\gamma}_3'$

η	$\bar{h}_1' = \frac{1}{2} h_1'$	$\bar{h}_2' = \frac{1}{2} h_2'$	$\bar{\gamma}_1' = \frac{1}{2} \gamma_1'$	$\bar{\gamma}_2' = 5\gamma_2'$	$\bar{\gamma}_3' = \frac{1}{2} \gamma_3'$
0	0	0	0	0	0
.1	.10000	.10008	.00001	.00000	.00500
.2	.20007	.20013	.00013	.00000	.02001
.3	.30034	.30068	.00067	.00000	.04506
.4	.40106	.40213	.00212	.00000	.08026
.5	.50259	.50520	.00516	.00002	.12578
.6	.60536	.61076	.01061	.00007	.18193
.7	.70988	.71988	.01947	.00025	.24915
.8	.81674	.83381	.03277	.00071	.32804
.9	.92661	.95394	.05161	.00179	.41937
1.0	1.04015	1.08181	.07705	.00404	.52410
1.1	1.15810	1.21906	.11004	.00840	.64336
1.2	1.28116	1.36747	.15137	.01622	.77847
1.3	1.41001	1.52883	.20162	.02941	.93087
1.4	1.54531	1.70501	.26112	.05069	1.10217
1.5	1.68773	1.89788	.32993	.08322	1.29406
1.6	1.83756	2.10934	.40787	.13100	1.50833
1.7	1.99549	2.34123	.49453	.19864	1.74679
1.8	2.16180	2.59540	.58937	.29123	2.01130
1.9	2.33678	2.87363	.69174	.41419	2.30370
2.0	2.52066	3.17768	.80096	.57312	2.62582
2.1	2.71362	3.50927	.91640	.77359	2.97948
2.2	2.91576	3.87005	1.03750	1.02104	3.36646
2.3	3.12716	4.26166	1.16379	1.32062	3.78850
2.4	3.34788	4.68571	1.29499	1.67717	4.24735
2.5	3.57795	5.14377	1.43120	2.09522	4.74468
2.6	3.81739	5.63740	1.57154	2.57897	5.28221
2.7	4.06621	6.16814	1.71615	3.13239	5.86159
2.8	4.32441	6.73752	1.86509	3.75920	6.48449
2.9	4.59199	7.34705	2.01847	4.46299	7.15257
3.0	4.86897	7.99825	2.17627	5.24721	7.86747

TABLES OF THE FUNCTIONS $\bar{h}_1'$, $\bar{h}_2'$, $\bar{\gamma}_1'$, $\bar{\gamma}_2'$, $\bar{\gamma}_3'$ - Concluded

η	$\bar{h}_1' = \frac{1}{2} h_1'$	$\bar{h}_2' = \frac{1}{2} h_2'$	$\bar{\gamma}_1' = \frac{1}{2} \gamma_1'$	$\bar{\gamma}_2' = 5\gamma_2'$	$\bar{\gamma}_3' = \frac{1}{2} \gamma_3'$
3.1	5.15533	8.69262	2.33856	6.75936	8.63084
3.2	5.45109	9.43165	2.50542	7.07063	9.44433
3.3	5.75623	10.21685	2.67692	8.11634	10.30959
3.4	6.07076	11.04968	2.85318	9.25549	11.22824
3.5	6.39467	11.93166	3.03426	10.49157	12.20193
3.6	6.72898	12.86426	3.22028	11.82765	13.23229
3.7	7.07068	13.84896	3.41134	13.26687	14.32096
3.8	7.42276	14.88724	3.60754	14.81237	15.46958
3.9	7.78424	15.98058	3.80898	16.46731	16.67977
4.0	8.15510	17.13046	4.01578	18.23482	17.95318
4.1	8.53535	18.33834	4.22805	20.11787	19.29142
4.2	8.92499	19.60571	4.44588	22.11970	20.69615
4.3	9.32402	20.93404	4.66940	24.24337	22.16899
4.4	9.73244	22.32479	4.89872	26.49183	23.71156
4.5	10.15025	23.77944	5.13396	28.86825	25.32550
4.6	10.57745	25.29946	5.37522	31.37567	27.01244
4.7	11.01403	26.88631	5.62262	34.01707	28.77401
4.8	11.46001	28.54147	5.87628	36.79559	30.61184
4.9	11.91537	30.26640	6.13632	39.71421	32.52756
5.0	12.38012	32.06257	6.40286	42.77598	34.52279
5.1	12.85427	33.93145	6.67600	45.98397	36.59918
5.2	13.33780	35.87449	6.95588	49.34105	38.75833
5.3	13.83072	37.89317	7.24262	52.85038	41.00189
5.4	14.33302	39.98895	7.53632	56.51492	43.33147
5.5	14.84472	42.16229	7.83825	60.32919	45.74825
5.6	15.36581	44.41767	8.14514	64.32172	48.25525
5.7	15.89628	46.75353	8.46050	68.46992	50.85269
5.8	16.43615	49.17236	8.78330	72.78548	53.54268
5.9	16.98540	51.67560	9.11369	77.27121	56.32683
6.0	17.54404	54.26473	9.45178	81.93026	59.20677

APPENDIX II

REMARKS CONCERNING H. GÖRTLER'S DIFFERENCE METHOD IN THE CASE OF SUCTION

The method developed in the present report permits further calculation of a boundary-layer flow in spite of the influence of a discontinuity of the value $v(x,0)$ of the v-velocity component at the wall. For the remaining flow regions with continuous $v(x,0)$, one may then use without difficulty one of the customary methods for boundary-layer calculation. (See section 2.) In our examples - as far as no exact solutions were available - we always used the difference method developed by H. Görtler [10]. For the flow without suction, consideration of the improvements indicated by H. Witting [19] proved to be very favorable. These improvements amount to a more exact treatment of the region near the wall and produce, particularly in the proximity of the separation point, a significant increase in accuracy. For the calculation of the flow with suction, in contrast, use of those improvements is no longer profitable - first, because they become then somewhat more troublesome, and second, because in this case the original method of Görtler works much more satisfactorily, in the proximity of the wall as well. The reason for the latter fact lies especially in that the u-velocity profiles have, in the case of suction, a "fuller" form, that is, they correspond for instance to velocity profiles without suction as appear far ahead of the separation point.

In flows with suction, other small variations of Görtler's difference method, instead of Witting's improvements, have proved extraordinarily favorable. We shall describe these variations below. We refer to the original report quoted [10] and can therefore be brief.

In Görtler's difference method the boundary-layer equations in their dimensionless form

$$\left.\begin{aligned} u\,u_x + v\,u_y &= u_\infty u_\infty' + u_{yy} \\ u_x + v_y &= 0 \end{aligned}\right\} \tag{1}$$

with the boundary conditions

$$u(x,0) = v(x,0) = 0 \qquad\qquad u(x,\infty) = u_\infty(x) \tag{1a}$$

which we change into

$$u(x,0) = 0 \qquad v(x,0) = -v_0(x) \qquad u(x,\infty) = u_\infty(x) \tag{1b}$$

are replaced by finite difference equations. For this purpose we introduce, first, in the half plane $y \geqq 0$ a grid system

$$\begin{array}{lll} x_i = x_0 + ih & (i=0,1,\ldots) & (h > 0) \\ y_k = kl & (k=0,1,\ldots) & (l > 0) \end{array}$$

and write for abbreviation

$$u_{ik} = u(x_i,y_k) \qquad v_{ik} = v(x_i,y_k) \qquad (u_\infty u_\infty')_i = u_\infty(x_i)u_\infty'(x_i) \tag{2}$$

Görtler approximates the derivatives u_x, u_y, and u_{yy} by the finite expressions

$$u_x(x_i,y_k) = \frac{\Delta_{ik}}{2h} = \frac{1}{2h}\left(u_{i+1,k} - u_{i-1,k}\right) \tag{3a}$$

$$u_y(x_i,y_k) = \frac{\nabla_{ik}}{2l} = \frac{1}{2l}\left(u_{i,k+1} - u_{i,k-1}\right) \qquad (k \geqq 1) \tag{3b}$$

$$\begin{aligned} u_{yy}(x_i,y_k) &= \frac{\nabla^2_{ik}}{4l^2} = \frac{1}{4l^2}\left(\nabla_{i,k+1} - \nabla_{i,k-1}\right) \qquad (k \geqq 2) \\ &= \frac{1}{4l^2}\left(u_{i,k+2} - 2u_{i,k} + u_{i,k-2}\right) \end{aligned} \tag{3c}$$

These approximations by means of quotients of "alternating" differences have, compared to those by means of the usual difference quotients, the advantage of greater accuracy. On the other hand, they involve various difficulties.

If the approximate equation (3a) is used, every two successive function values u_{ik} and $u_{i+1,k}$ are connected only indirectly through the differential equation. Thereby it becomes evident in the calculation that the two sequences $u_{-1,k}$, $u_{1,k}$, $u_{3,k}$, . . . and $u_{0,k}$, $u_{2,k}$, $u_{4,k}$, . . . have for constant k, a smooth course in every case, but deviate more and more from each other, with increasing first subscript. This "splitting-up" of the solution requires a consecutive smoothing of the calculation results. We refer, for this, to the detailed considerations in H. Witting [19]. This difficulty is eliminated to a great extent if, instead of (3a), approximation by ordinary differences

$$u_x(x_i,y_k) = \frac{d_{ik}}{h} = \frac{1}{h}\left(u_{i+1,k} - u_{ik}\right) \tag{3a*}$$

is used. In so doing, one has to make allowance (for equal step interval) for a slight increase in the rounding-off error; however - as H. Witting shows in [20] - this is amply compensated by an essential increase in stability of the method with respect to small numerical disturbances. The effect of this in the calculation is precisely a strong reduction of the splitting-up mentioned above.

A further unfavorable consequence of the approximations (3) is that they leave the derivatives

$$u_y(x_i,0) \qquad u_{yy}(x_i,0) \qquad u_{yy}(x_i,l) \tag{4}$$

undetermined. This is not too bad for $u_y(x_i,0)$ and $u_{yy}(x_i,0)$ since these values do not enter any farther into the present method. But, on the other hand, one requires for $u_{yy}(x_i,l)$ - because of the proximity to the wall - a finite expression which must be more exact. For flows without suction one has here the great advantage that on the basis of the first wall restrictions

$$u_{yy}(x_i,0) = -(u_\infty u_\infty')_i$$

the derivative $u_{yy}(x_i,0)$ is known. If we set therefore

$$\nabla_{i0}^2 = 4l^2 u_{yy}(x_i,0) = -4l^2(u_\infty u_\infty')_i$$

the missing second difference $\nabla_{i1}^2 = 4l^2 u_{yy}(x_i,l)$ may now be obtained for instance by cubic interpolation in the differences of the second degree which leads to

$$\nabla_{i1}^2 = \frac{11}{18}\nabla_{i0}^2 + \frac{1}{2}\nabla_{i2}^2 - \frac{1}{9}\nabla_{i3}^2$$

Thus one obtains finally also $\nabla_{i0} = 2l u_y(x_i,0)$ from

$$\nabla_{i0} = \nabla_{i2} - \nabla_{i1}^2$$

For flows with suction the method for determining ∇_{i1}^2 described just now is not applicable since the first wall restriction now reads

$$u_{yy}(x_i,0) = -(u_\infty u_\infty')_i + u_y(x_i,0)v_{i0}$$

thus the quantities on both sides are unknown. Therefore one would be dependent on obtaining a finite expression for ∇_{i1}^2 by extrapolation; however, it is hardly possible to attain a satisfactory accuracy. As experience shows, a very widely varying quality of approximation has a much more unfavorable effect on a method than an approximation of lower degree which is uniform throughout. Accordingly, it lends itself, similarly to the procedure for u_x, to approximate the derivatives u_{yy} also by ordinary difference quotients

$$u_{yy}(x_i,y_k) = \frac{\delta_{ik}}{l^2} = \frac{\delta_{ik+1} - \delta_{ik}}{l^2} = \frac{1}{l^2}(u_{i,k+1} - 2u_{ik} + u_{i,k-1}) \qquad (3c^*)$$

$(k \geqq 1)$ since thereby $u_{yy}(x_i, l)$, too, is determined. This approximation formula (3c*) has still another advantage compared to (3c): (3c*) signifies geometrically for every x_i the approximation of the curvature of the curve $u(x_i,y_k)$ by means of the circle through the points $u_{i,k+1}$, u_{ik}, $u_{i,k-1}$, whereas in the case (3c) for the same problem the points $u_{i,k+2}$, u_{ik}, $u_{i,k-2}$ are used which lie farther apart. For more pronounced variations in curvature which still occur for instance at a considerable distance behind the jumplike suction start, this has evidently very unfavorable effects.

A perfectly analogous development of the difference method as indicated by Görtler is now possible. For this purpose we evaluate first the integral

$$v(x,y) = v(x,0) - \int_0^y u_x(x,y)\, dy$$

which follows from the continuity equation for the velocity component v, with consideration of the boundary conditions (1b) and the approximation (3c*), with the aid of the trapezoidal rule

$$v_{ik} = v_{i0} - \frac{l}{2h} d_{ik} - \frac{l}{h} \sum_{\nu=1}^{k-1} d_{i\nu} \qquad (5)$$

If (2), (3a*), (3b), (3c*), and (5) are then introduced into (1), there results, for solution with respect to the unknown d_{ik}

$$d_{ik} = \frac{2h(u_\infty u_\infty')_i + \frac{2h}{l^2}\delta_{ik}^2 + \nabla_{ik}\left(\sum_{\nu=1}^{k-1} d_{i\nu} - \frac{h}{l} v_{i0}\right)}{2\,u_{ik} - \frac{1}{2}\nabla_{ik}} \qquad (6)$$

For carrying out a calculation step - thus the numerical calculation of the d_{ik} for constant i - one uses suitably for instance the following scheme

$l = \ldots$ $h = \ldots$ $2h(u_\infty u_\infty')_i = \ldots$ $-\frac{h}{l} v_{i0} = \ldots$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
y_k	u_{ik}	∇_{ik}	δ^2_{ik}	$2u_{ik}-0.5\nabla_{ik}$	$-\frac{h}{l} v_{i0} + \sum_{\nu=1}^{k-1} d_{i\nu}$	d_{ik}	$u_{i+1,k}$
0	0				$-\frac{h}{l} v_{i0}$	0	0
l	u_{i1}	∇_{i1}	δ^2_{i1}	$2u_{i1}-0.5\nabla_{i1}$	$-\frac{h}{l} v_{i0}$	d_{i1}	$u_{i1} + d_{i1}$
$2l$	u_{i2}	∇_{i2}	δ^2_{i2}	$2u_{i2}-0.5\nabla_{i2}$	$-\frac{h}{l} v_{i0} + d_{i1}$	d_{i2}	$u_{i2} + d_{i2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

One calculates therefore first in every operation on the machine

$$(3) = \nabla_{ik} = u_{i,k+1} - u_{i,k-1}$$

$$(4) = \delta^2_{ik} = u_{i,k+1} - 2\,u_{ik} + u_{i,k-1}$$

$$(5) = N_{ik} = 2\,u_{ik} - 0.5\nabla_{ik}$$

Then the two last columns may be filled in line by line. For every k, one may obtain the value

$$(7) = d_{ik} = \frac{(3)(6) + \left(2h/l^2\right)(4) + 2h(u_\infty u_\infty')_i}{(5)}$$

again in one operation. For performance of such a calculation step with about 20 profile points y_k, an experienced calculator requires approximately half an hour.

Regarding the step intervals h and l we have found by experience that, for $\frac{2h}{l^2}$, numerical values up to at most 1.0 are most favorable. For $\frac{2h}{l^2} > 1$, in contrast, soon a certain accumulation of error becomes noticeable for the values near the wall of d_{ik}. Small step intervals l require, therefore, also small h. As one soon finds out, a distinct optimum exists for every example. A phenomenon of splitting-up of the type described above will occur generally only to a very limited extent for u_{i1}, u_{i2}, and possibly for u_{i3}. For its elimination, the smoothing rule given by H. Witting

$$d^*_{ik} = 0.25\, d_{i+1,k} - 0.5\, d_{ik} + 0.25\, d_{i-1,k} \tag{7}$$

is very useful. One first calculates (which requires little expenditure) the three first values $d_{i+1,k}$ from the $(i + 1)^{th}$ step, determines with these the improved d^*_{ik} according to (7) and repeats the same procedure iteratively, if necessary. This smoothing of the d-values near the wall may be performed mechanically perhaps at every third or fourth step and produces then a very satisfactory variation of all sequences of values.

After the calculation has been carried out in the manner described above, it remains to calculate the values $u_y(x_i,0)$ of the wall-shear stress, not required until now. One may choose various methods. If only a first survey is desired, one will put for instance

$$u_y(x_i,0) = \frac{\delta_{i2} - \delta^2_{i1}}{l} = \frac{1}{l}\,(u_{i3} - 2u_{i2} + 2u_{i1})$$

A better finite expression is obtained, in a manner analogous to H. Witting's, by means of the formulation

$$\delta_{i0} = K\Theta_i + \sum_{\nu=1}^{5} A_\nu\, u_{i\nu} \qquad (\Theta_i = -4l^2(u_\infty u'_\infty)_i)$$

if the latter is expanded according to Taylor and the first two wall restrictions

$$u_{yy}(x,0) = -u_\infty u'_\infty - u_y(x,0)\; v_0(x)$$

$$u_{yyy}(x,0) = -u_{yy}(x,0)\; v_0(x)$$

are taken into consideration. By comparison of the coefficients, a system of four linear equations for the six unknown coefficients originates. It has proved expedient, because of the bad error propagation at the wall, simply to put $A_1 = A_2 = 0$, according to H. Witting's method. Then the finite expression for δ_{i0} reads finally

$$\delta_{i0} = \frac{1}{N}\left[-(21150 + 9000\,\overline{v}_i)\Theta_i + 80000\,u_{i3} - 50625\,u_{i4} + 10368\,u_{i5}\right]$$

with

$$N = 89740 + 84600\,\overline{v}_i + 36000\,\overline{v}_i^2$$

and

$$\overline{v}_i = -l v_0(x_i)$$

This corresponds, for $\overline{v}_i = 0$, exactly to the formula given by H. Witting for the flow without suction.

APPENDIX III

ON A GENERALIZED KIND OF ASYMPTOTIC SERIES

In section 5, an asymptotic series of the form

$$F(x) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{k} a_{kl} x^k \ln^{k-l} x \tag{1}$$

for $x \to 0$ occurred. We have still to show that one can calculate with this series in exactly the same way as with ordinary asymptotic expansions. Below, we shall produce this proof, but directly for the considerably more general series of the form

$$\sum_{n=0}^{\infty} a_n \Phi_n(x) \tag{2}$$

for $x \to x_0$ with functions $\Phi_n(x)$ corresponding to

<u>Definition 1</u>: Let an infinite sequence of functions $\Phi_n(x)$ $(n = 0,1, 2, \ldots)$ be prescribed, with the properties

(a) All $\Phi_n(x)$ are continuous and different from zero in a half-open interval J of the form $x_1 \leqq x < x_0$ or $x_0 < x \leqq x_2$ - for which $x_0 = \pm\infty$ also is permissible

(b) At the excluded boundary point x_0 of J, let the limiting value $\lim\limits_{x \to x_0;\ x \in J} \Phi_n(x)$ exist for every n.

(c) For all n there is always:

$$\lim_{\substack{x \to x_0 \\ x \in J}} \frac{\Phi_{n+1}(x)}{\Phi_n(x)} = 0 \tag{3}$$

Such a sequence of functions we shall call a Φ-sequence over J with the limiting point x_0, and denote it by $\{\Phi_n\}_{J,x_0}$.

A very simple example of such a Φ-sequence is formed by the functions

$$\Phi_n(x) = x^n \qquad (n=0,1,2,\ldots) \qquad (4)$$

for $x \to 0$ where J is, for instance, the interval $0 < x \leqq x_2$. A similar example is, of course, the case

$$\Phi_n(x) = \frac{1}{x^n} \qquad (n=0,1,2,\ldots) \qquad (5)$$

for $x \to \infty$ in $(0 <) x_1 \leqq x < \infty$. The functions

$$\Phi_n(x) = x^k \ln^{k-l} x \qquad (k=0,1,\ldots;\ l=0,1,\ldots) \qquad (6)$$

occurring in the series (1), with $n = \binom{k+1}{2} + l$ likewise represent a Φ-sequence for $x \to 0$ in $0 < x \leqq x_2$.

In all Φ-sequences, there results from the properties (b) and (c) immediately

$$\lim_{x \to x_0} \Phi_n(x) = 0 \qquad (n=1,2,\ldots) \quad (x \epsilon J) \qquad (7a)$$

whereas nothing is stated concerning the limiting value of $\Phi_0(x)$. In what follows, we shall always assume

$$\lim_{x \to x_0} \Phi_0(x) \neq 0 \qquad (x \epsilon J) \qquad (7b)$$

This does not represent any restriction of the generality; if $\lim_{x \to x_0} \Phi_0(x) = 0$, we simply consider the Φ-sequence in the extended form $\Phi_0{}^*(x) \equiv 1$, $\Phi_n{}^*(x) \equiv \Phi_{n-1}(x)$ $(n \geqq 1)$.

Furthermore, for every Φ-sequence there applies the rule that any finite number of its functions $\Phi_n(x)$ are always linearly independent in the entire interval J. Let us assume that there exists a relation of the form

$$\sum_{\substack{\text{finite number} \\ \text{of } n}} c_n \Phi_n(x) \equiv 0 \qquad \left(\sum_n |c_n| > 0 \right)$$

in which for instance n_1 is to be the smallest subscript, with $c_{n_1} \neq 0$. Then we obtain, after division by $\Phi_{n_1}(x)$ for $x \to x_0$, a contradiction since the left side of the equation tends toward $c_{n_1} \neq 0$, because of (3).

With any arbitrary Φ-sequence, we can now form series of the form (2) which we shall explain, in analogy to the ordinary asymptotic expansions, as follows:

Definition 2: Let $\left\{\Phi_n\right\}_{J,x_0}$ be a Φ-sequence. A series of the form $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ which need nowhere converge is called an asymptotic Φ-sequence for $x \to x_0$ $(x \epsilon J)$ of the function $F(x)$ defined in J when for every m

$$\lim_{\substack{x \to x_0 \\ x \epsilon J}} \left[F(x) - \sum_{n=0}^{m} a_n \Phi_n(x) \right] \frac{1}{\Phi_m(x)} = 0 \tag{8}$$

For this we write abbreviately

$$F(x) \sim \sum_{n=0}^{\infty} a_n \Phi_n(x) \tag{9}$$

Corresponding to our above examples, this definition contains the ordinary asymptotic series and also the expansion (1) as special cases.

On the basis of our assumption (7b) made for all Φ-sequences, one may assume, in every Φ-series, $\Phi_0(x) \equiv 1$ as normalized. This signifies only a transition to the Φ-expansion

$$\frac{F(x)}{\Phi_0(x)} \sim \sum_{n=0}^{\infty} a_n \Phi_n^*(x)$$

with the Φ-sequence

$$\left\{\Phi_n{}^* = \frac{\Phi_n}{\Phi_0}\right\}_{J,x_0}$$

We must now investigate how we may calculate with the general asymptotic Φ-series. The Φ-sequence used, $\left\{\Phi_n\right\}_{J,x_0}$, will be assumed to be always the same and a priori fixed so that all occurring arguments x lie in the same interval J and all limiting processes $x \to x_0$ are referred to the same x_0.

First of all we find

Theorem 1: A function $F(x)$ defined in J can be represented for $x \to x_0$ by at most one asymptotic Φ-series.

Proof: From (8) there follows immediately

$$\lim_{x \to x_0} \left[F(x) - \sum_{n=0}^{m=1} a_n\Phi_n(x)\right] \frac{1}{\Phi_m(x)} = a_m \qquad (m = 0,1,\ldots) \qquad (10)$$

that is

$$\left.\begin{aligned}
&\lim_{x \to x_0} \frac{F(x)}{\Phi_0(x)} = a_0 \\
&\lim_{x \to x_0} \left(F(x) - a_0\Phi_0(x)\right) \frac{1}{\Phi_1(x)} = a_1 \\
&\lim_{x \to x_0} \left(F(x) - a_0\Phi_0(x) - a_1\Phi_1(x)\right) \frac{1}{\Phi_2(x)} = a_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (11)$$

Only when all these boundary values exist, $F(x)$ has, for $x \to x_0$, a Φ-series over J, and this series is then uniquely determined.

Theorem 2: Assume that it is possible to represent the functions $F(x)$ and $G(x)$ by the Φ-series

$$F(x) \sim \sum_{n=0}^{\infty} a_n \Phi_n(x) \qquad G(x) \sim \sum_{n=0}^{\infty} b_n \Phi_n(x) \tag{12}$$

Then, with arbitrary constants α, β the linear combination $\alpha F(x) + \beta G(x)$ also has, for $x \to x_0$, a Φ-series over J, namely

$$\alpha F(x) + \beta G(x) \sim \sum_{n=0}^{\infty} \left(\alpha a_n + \beta b_n\right) \Phi_n(x) \tag{13}$$

Proof: From the boundary values valid for every m

$$\lim_{x \to x_0} \left[F(x) - \sum_{n=0}^{m} a_n \Phi_n(x) \right] \frac{1}{\Phi_m(x)} = 0$$

$$\lim_{x \to x_0} \left[G(x) - \sum_{n=0}^{m} b_n \Phi_n(x) \right] \frac{1}{\Phi_m(x)} = 0$$

follows directly by multiplication by α and β and following addition the assertion

$$\lim_{x \to x_0} \left[\left(\alpha F(x) + \beta G(x)\right) - \sum_{n=0}^{m} \left(\alpha a_n + \beta b_n\right) \Phi_n(x) \right] \frac{1}{\Phi_m(x)} = 0$$

The multiplication of Φ-series requires a further presupposition which guarantees that the range of the functions $\Phi_n(x)$ is closed with respect to the product formation. More accurately we say

Definition 3: A Φ-sequence $\left\{\Phi_n\right\}_{J, x_0}$ will be called multiplicative when for every pair of subscripts k, l exactly a subscript $n = n(k, l)$ exists so that in the entire J

$$\Phi_k(x) \Phi_l(x) \equiv \Phi_n(x)$$

is valid.

The Φ-sequences (4), (5), and (6) of our three examples are evidently all multiplicative.

L-401

If an arbitrary multiplicative Φ-sequence is prescribed, a multiplication table for it may be set up in a known manner. For our example (6), this table starts for instance with

1	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	. . .
Φ_1	Φ_3	Φ_4	Φ_6	Φ_7	. . . .	
Φ_2	Φ_4	Φ_5	Φ_7	Φ_8	. . . .	
Φ_3	Φ_6	Φ_7	Φ_{10}	Φ_{11}	. . . .	
Φ_4	Φ_7	Φ_8	Φ_{11}	Φ_{12}	. . . .	
Φ_5	. . . .					

In a single line or column of such a multiplication table, every Φ_n-funtion may, of course, appear only once at most. However, no statement is yet made regarding the question, how many times this Φ_n for fixed n occurs in the table altogether. Such a statement is contained only in the following

<u>Auxiliary theorem 1</u>: In a multiplicative Φ-sequence $\{\Phi_n\}_{J,x_0}$ there applies for the subscript $n = n(k,l)$ of the product $\Phi_n = \Phi_k\Phi_l$ of any two functions Φ_k and Φ_l always

$$n(k,l) \geqq k + l \tag{14}$$

<u>Proof</u>: The assertion of the theorem signifies that for fixed n the function Φ_n in the multiplication table of the sequence can never appear outside the triangle shown.

$$\begin{array}{llll}
1 & \Phi_1 & \cdots & \Phi_n \\
\Phi_1 & & \cdot & \\
\vdots & \cdot & & \\
\Phi_n & & &
\end{array}$$

In order to understand this, we consider first for fixed k two products

$$\Phi_k\Phi_{l_1} = \Phi_{n_1} \qquad \Phi_k\Phi_{l_2} = \Phi_{n_2}$$

If for them $l_1 < l_2$ applies, there must also always be $n_1 < n_2$. This follows from

$$\Phi_k \Phi_{l_1} \Phi_{l_2} = \Phi_{n_2} \Phi_{l_1} = \Phi_{n_1} \Phi_{l_2}$$

thus

$$\frac{\Phi_{l_2}}{\Phi_{l_1}} = \frac{\Phi_{n_2}}{\Phi_{n_1}}$$

If $n_2 < n_1$ is assumed, the left side of this equation tends toward zero for $x \to x_0$ $(x \epsilon J)$ because of the property (c) of the Φ-sequences; the right side, in contrast, tends toward infinity, for the same reason. The case $n_2 = n_1$ likewise results in a contradiction since then the right side is constant 1.

Let us now assume - for proof of the theorem itself - that a product $\Phi_k \Phi_l = \Phi_n$ with $k + l > n$ exists. This product then lies in the multiplication table outside the triangle drawn above. On the line of Φ_k there exist, up to the Φ_n, exactly l-1 intermediate places. Corresponding to our auxiliary consideration, however, only the $n - k - 1$ $(< l - 1)$ functions $\Phi_{k+1}, \ . \ . \ . \ , \ \Phi_{n-1}$ can be at these places. This is a contradiction; therefore, (14) must always be correct.

From the theorem just proved it follows that, for every fixed n, the number $p = p(n)$ of the products $\Phi_k \Phi_l = \Phi_n$ can be at most $n + 1$. We order these p products in the sequence

$$\Phi_{k_1} \Phi_{l_1}, \ . \ . \ . \ , \ \Phi_{k_p} \Phi_{l_p}$$

with $0 = k_1 < k_2 < \ . \ . \ . < k_{p-1} < k_p = n$ and $0 \leqq l \leqq n - k_\nu$ $(\nu = 1, \ . \ . \ . \ , p)$. We assume the pairs of subscripts (k_ν, l_ν) in our prescribed Φ-sequence as once for all determined from the multiplication table for every n.

Now applies

Theorem 3: Assume it possible to represent the functions $F(x)$ and $G(x)$ for $x \to x_0$ by the Φ-series (12) whose Φ-sequence $\left\{\Phi_n\right\}_{J,x_0}$ is multiplicative. Then the product $F(x)G(x)$, too, has for $x \to x_0$

a Φ-series over J, namely

$$F(x)G(x) \sim \sum_{n=0}^{\infty} c_n \Phi_n(x)$$

With the pairs of subscripts $(k_1, l_1), \ldots, (k_{p(n)}, l_{p(n)})$ introduced above one has here

$$c_n = a_{k_1} b_{l_1} + \cdots + a_{k_{p(n)}} b_{l_{p(n)}} \tag{15}$$

Proof: According to the definition of the Φ-series we may set for fixed m

$$F(x) \sim \sum_{n=0}^{m} a_n \Phi_n(x) + \underline{\epsilon}(x) \Phi_m(x)$$

$$G(x) \sim \sum_{n=0}^{m} b_n \Phi_n(x) + \eta(x) \Phi_m(x)$$

where $\underline{\epsilon}(x)$ and $\eta(x)$ are certain functions over J which, for $x \to x_0$, tend toward zero. We now multiply these two finite sums and then order according to the Φ_n. This results in

$$F(x)G(x) = \sum_{n=0}^{M} c_n^* \Phi_n(x) + \sum_{n=0}^{m} \left[a_n \eta(x) + b_n \underline{\epsilon}(x)\right] \Phi_n \Phi_m$$

where $M \geqq 2m$ is a certain fixed number and where the $c_0^*, c_1^*, \ldots, c_m^*$ evidently must be exactly equal to the corresponding c_n according to (15). Thus the above equation becomes

$$\left[F(x)G(x) - \sum_{n=0}^{m} c_n \Phi_n(x)\right] \frac{1}{\Phi_n(x)} = \sum_{n=m+1}^{M} c_n^* \frac{\Phi_n(x)}{\Phi_m(x)} +$$

$$\sum_{n=0}^{m} \left[a_n \eta(x) + b_n \underline{\epsilon}(x)\right] \Phi_n(x)$$

that is, because of (3) and $\lim\limits_{x \to x_0} \underline{\epsilon}(x) = \lim\limits_{x \to x_0} \eta(x) = 0$, directly, as asserted

$$\lim_{x \to x_0} \left[F(x)G(x) - \sum_{n=0}^{m} c_n\Phi_n(x) \right] \frac{1}{\Phi_m(x)} = 0$$

By repeated application of the theorems 2 and 3 there follows immediately

<u>Theorem 4</u>: If each of the functions $F_1(x), F_2(x), \ldots, F_s(x)$ has for $x \to x_0$, an asymptotic Φ-expansion whose Φ-sequence $\{\Phi_n\}_{J,x_0}$ is multiplicative, and if $g(z_1, \ldots, z_s)$ signifies any polynomial of the variable $z_1, \ldots, z_s$, the function

$$F(x) = g(F_1(x), F_2(x), \ldots, F_s(x))$$

also has, for $x \to x_0$, an asymptotic Φ-series over J. This series is obtained - exactly as if all expansions were absolutely convergent - by formal calculation and following reordering according to the functions Φ_n.

Beyond this, there applies now even the following:

<u>Theorem 5</u>: If $g(z) = \sum\limits_{\mu=0}^{\infty} \alpha_\mu z^\mu$ is a power series with the positive convergence radius r, and if $F(x)$ has the asymptotic Φ-expansion

$$F(x) \sim \sum_{n=0}^{\infty} a_n\Phi_n(x) \qquad (x \to x_0)$$

where $\Phi_0(x) \equiv 1$ and $|a_0| < r$, the function

$$G(x) = g(F(x)) \tag{16}$$

too, may be represented for $x \to x_0$, by an asymptotic Φ-series. This series is again calculated exactly as if the series $\sum\limits_{n=0}^{\infty} a_n\Phi_n(x)$ were convergent in J.

Proof: With $F(x) = a_0 + f(x)$ there is first

$$g(F) = g(a_0 + f) = \sum_{n=0}^{\infty} \beta_k f^k \tag{17}$$

if one puts for abbreviation

$$\beta_k = \frac{1}{k!} g^{(k)}(a_0)$$

According to assumption, this series converges for $|f(x)| < r - |a_0|$, thus because of $\lim_{x \to x_0} f(x) = 0$ for all x from a half-open interval J^* explained by $x \epsilon J$, $|x - x_0| \leqq \delta$ with sufficiently small $\delta > 0$. Thereby the function $G(x)$ is with certainty defined in J^*.

On the basis of theorem 4, there now follow from

$$f(x) \sim \sum_{n=1}^{\infty} a_n \Phi_n(x) \tag{18}$$

immediately the Φ-expansions

$$(f(x))^k \sim \sum_{n=k}^{\infty} a_n^{(k)} \Phi_n(x) \qquad (k=2,3,\dots) \tag{19}$$

The coefficients $a_n^{(k)}$ have well-determined values to be calculated according to our rules regarding the product-formation of asymptotic Φ-expansions - thus as for absolute convergence of (18). On the basis of the auxiliary theorem 1, the series (19) can of course, for $(f(x))^k$, start at the earliest with the term $a_k^{(k)} \Phi_k(x)$.

The expansions (19) must now be substituted into (17) and the obtained expression must be reordered formally - that is, again, as if the series (19) were absolutely convergent - according to the functions Φ_n. One then obtains an expansion

$$\sum_{n=0}^{\infty} A_n \Phi_n(x) \tag{20}$$

for which the coefficients can be calculated from

$$\left.\begin{aligned} A_0 &= \beta_0 \\ A_n &= \beta_1 a_n + \beta_2 a_n^{(2)} + \dots + \beta_n a_n^{(n)} \qquad (n \geqq 1) \end{aligned}\right\} \tag{21}$$

It remains to be shown that (20) is the asymptotic Φ-series of $G(x)$ for $x \to x_0$ over J^*. For this purpose we use for $f(x)$, $(f(x))^2$, . . . , $(f(x))^m$, for fixed m, the defining representations pertaining to (19)

$$(f(x))^k = \sum_{n=k}^{m} a_n^{(k)} \Phi_n(x) + \underline{\varepsilon}_k(x)\Phi_m(x)$$

with certain functions $\underline{\varepsilon}_k(x)$ for which $\lim_{x \to x_0} \underline{\varepsilon}_k(x) = 0$ $(k = 1, . . . , m)$ is valid. Thereby (17) becomes

$$G(x) = \beta_0 + \beta_1 \left[\sum_{n=1}^{m} a_n \Phi_n(x) + \underline{\varepsilon}_1(x)\Phi_m(x) \right] + \dots + \beta_m \Big(a_m^{(m)} \Phi_m(x) +$$

$$\underline{\varepsilon}_m(x)\Phi_m(x) \Big) + f^{n+1} \sum_{\mu=1}^{\infty} \beta_{n+\mu} f^{\mu}$$

thus, with consideration of (21)

$$G(x) - \sum_{n=0}^{m} A_n \Phi_n(x) = \Phi_m(x) \left[\beta_1 \underline{\varepsilon}_1(x) + \dots + \beta_m \underline{\varepsilon}_m(x) \right] +$$

$$f^{n+1} \sum_{\mu=1}^{\infty} \beta_{n+\mu} f^{\mu}$$

Since one must have, because of (19) and (3)

$$\lim_{x \to x_0} \frac{(f(x))^{n+1}}{\Phi_m(x)} = 0$$

and moreover evidently

$$\lim_{x \to x_0} \beta_1 \varepsilon_1(x) + \dots + \beta_m \varepsilon_m(x) = 0$$

$$\lim_{x \to x_0} \sum_{\mu=1}^{\infty} \beta_{n+\mu} f^{\mu} = \beta_{n+1}$$

are valid, the assertion follows.

An important special case of this theorem is obtained for

$$g(z) = \frac{1}{a_0 + z} = \frac{1}{a_0} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{a_0}\right)^n \qquad (a_0 \neq 0)$$

if for z the asymptotic Φ-series

$$f(x) = F(x) - a_0 \sim \sum_{n=1}^{\infty} a_n \Phi_n(x)$$

is substituted. Then the theorem states precisely that we may also divide by an asymptotic Φ-expansion, exactly as if it were absolutely convergent, if only the constant term is $a_0 \neq 0$.

Now we shall turn to the question under what conditions one can calculate from the Φ-series of a (continuous) function $F(x)$ a likewise continuous expansion for the integral

$$G(x) = \int_{x_0}^{x} F(\xi)\, d\xi \qquad (x \in J)$$

For this, it will certainly be required that - similarly to the case of multiplication - the integrations

$$\int_{x_0}^{x} \Phi_n(\xi)\, d\xi \qquad (x \in J) \qquad (22)$$

can always be carried out and do not lead out of the Φ-sequence used.

The existence of (22) is, for finite x_0, always guaranteed on the basis of the properties of the Φ-functions for all $x \epsilon J$ and every $n \geqq 0$. For $x_0 = \pm\infty$, in contrast, those integrals are improper and possibly in part even divergent - as shown by the example of the normalized $\Phi_0(x) \equiv 1$. In this case we know generally merely: when (22) converges for a subscript $n = M$, this is necessarily valid also for all $n \geqq M$. This follows directly with the aid of the known majorant principle of improper integrals if one only takes into consideration that from (3) for every $n \geqq M$ dissimilar terms of the form

$$\left|\Phi_n(x)\right| \leqq c_{nM}\left|\Phi_M(x)\right|$$

may be derived, with a constant c_{nM} valid in the entire J.*

In order to be able to make a rational assumption regarding the result of the integrations (22), we consider first our three examples (4), (5), and (6). For (4) and (6)

$$\int_0^x \xi^n d\xi = \frac{x^{n+1}}{n+1} \qquad (23)$$

is valid or, respectively,

$$\int_0^x \xi^k \ln^{k-l}\xi \, d\xi =$$

$$\sum_{\mu=0}^{k-l} (-1)^\mu \frac{(k-l)(k-l-1)..(k-l-(\mu-1))}{(k+1)^{\mu+1}} x^{k+1} \ln^{k-l-\mu} x \qquad (24)$$

In contrast, we find for (5), for $n \geqq 2$,

$$\int_x^\infty \xi^{-n} d\xi = \frac{1}{(n-1)x^{n-1}} \qquad (25)$$

whereas the two integrals $\int_x^\infty d\xi$ or $\int_x^\infty \xi^{-1} d\xi$ do not exist at all. In the two formulas (23) and (24) it is noteworthy that on the right side only Φ-functions having a larger subscript than the integrand occur.

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*Here we tacitly made use of the fact that the $\Phi_n(x)$ in J are all different from zero and that, therefore, in the entire J either always $\left|\Phi_n(x)\right| = \Phi_n(x)$ or $\left|\Phi_n(x)\right| = -\Phi_n(x)$.

This is a result valid generally for $x_0 \neq \pm\infty$. Let us assume, for a finite x_0 and an arbitrary fixed n

$$\int_{x_0}^{x} \Phi_n(\xi)d\xi = \sum_{\mu=n-N}^{j(n)} c_{n\mu}\Phi_\mu(x) \tag{26}$$

with $N > 0$ (that is, $n > 0$). Then the integral may be replaced, as is well known, by

$$\int_{x_0}^{x} \Phi_n(\xi)d\xi = \Phi_n(x^*)(x - x_0) \tag{27}$$

where x^* represents an intermediate value in the integration interval. Hence follows

$$\lim_{x \to x_0} \frac{\int_{x_0}^{x} \Phi_n(\xi)d\xi}{\Phi_n(x)} = \lim_{x \to x_0} \frac{\Phi_n(x^*)}{\Phi_n(x)}(x - x_0) = 0$$

whereas for the right side of (26), because of (3),

$$\lim_{x \to x_0} \frac{\sum_{\mu=n-N}^{j(n)} c_{n\mu}\Phi_\mu(x)}{\Phi_n(x)} = \sum_{\mu=n-N}^{j(n)} c_{n\mu} \lim_{x \to x_0} \frac{\Phi_\mu(x)}{\Phi_n(x)} = \infty$$

is valid. This is a contradiction as long as we do not set $N = 0$. For $x_0 = \pm\infty$ this result cannot be correct, of course, as shown by (25), because then the mean-value theorem (27) loses its validity.

Corresponding to what has been said until now, we shall agree:

Definition 4: Let $\left\{\Phi_n\right\}_{J,x_0}$ be a Φ-sequence with the boundary point x_0. We denote by M the smallest subscript for which the integral $\int_{x_0}^{x} \Phi_n(\xi)d\xi$ in J exists. For finite x_0, there must always be $M = 0$;

for $x_0 = \pm\infty$, let us assume the existence of a finite M. Then the Φ-sequence will be called integrable when for all subscripts $n \geqq M$ in the entire J

$$\int_{x_0}^{x} \Phi_n(\xi)d\xi = \sum_{\mu=n-M}^{j(n)} c_{n\mu}\Phi_\mu(x) \tag{28}$$

is valid, with finite summation limits $j(n)$ and certain coefficients $c_{n\mu}$, uniquely determined because of the linear independence of the Φ_μ.

Accordingly, all three examples (4), (5), and (6) are, of course, integrable Φ-sequences.

We can now show:

<u>Theorem 6</u>: Let us assume that the function $F(x)$, continuous in J, can be represented, for $x \to x_0$, by the Φ-series

$$F(x) \sim \sum_{n=0}^{\infty} a_n\Phi_n(x) \tag{29}$$

with the Φ-sequence $\left\{\Phi_n\right\}_{J,x_0}$ assumed to be integrable. Then the function

$$G(x) \quad \int_{x_0}^{x}\left(F(\xi) - \sum_{n=0}^{M-1} a_n\Phi_n(\xi)\right)d\xi \tag{30}$$

also defined in J and has there for $x \to x_0$ an asymptotic Φ-series

$$G(x) \sim \sum_{n=0}^{\infty} A_n\Phi_n(x) \tag{31}$$

The coefficients A_n are obtained from the Φ-expansion $\sum_{\eta=M}^{\infty} a_n\Phi_n(x)$ of the integrand of (30) by term-by-term integration and following reordering with respect to the Φ_n, exactly as if (29) were absolutely convergent.

<u>Proof</u>: The existence of the function $G(x)$ in J follows, for finite x_0, directly from the continuity of $F(x)$. For $x_0 = \pm\infty$, in contrast, it results, again with the aid of the majorant principle, from the dissimilar terms

$$\left| F(x) - \sum_{n=0}^{M-1} a_n \Phi_n(x) \right| = \left| \left(a_M + \underline{\varepsilon}_M(x)\right) \Phi_M(x) \right| \leqq c_M \left| \Phi_M(x) \right|$$

since $\int_{x_0}^{x} \Phi_M(\xi)d\xi$ was to converge.

We now use for (29) the representation

$$F(x) - \sum_{n=0}^{M-1} a_n \Phi_n(x) = \sum_{n=M}^{M+m} a_n \Phi_n(x) + \underline{\varepsilon}(x) a_{m+M} \Phi_{m+M}(x) \tag{32}$$

with arbitrary but fixed m and a function $\underline{\varepsilon}(x)$ defined in J which for $x \to x_0$ tends toward zero. Thereby $G(x)$ becomes

$$G(x) = \sum_{n=M}^{M+m} a_n \sum_{\mu=n-M}^{j(n)} c_{n\mu} \Phi_\mu(x) + \int_{x_0}^{x} \underline{\varepsilon}(\xi) \Phi_{M+m}(\xi) d\xi$$

thus after reordering

$$G(x) = \sum_{n=0}^{j(M+m)} A_n^* \Phi_n(x) + \int_{x_0}^{x} \underline{\varepsilon}(\xi) \Phi_{m+M}(\xi) d\xi$$

Evidently the coefficients $A_0^*, \ldots, A_m^*$ must now be exactly equal to the corresponding A_n from (31) since they can no longer be affected by any term of the series (29) with a subscript larger than $m + M$. Thereby one has

$$\left[G(x) - \sum_{n=0}^{m} A_n \Phi_n(x) \right] \frac{1}{\Phi_m(x)} = \sum_{n=m+1}^{j(m+n)} A_n^* \frac{\Phi_n(x)}{\Phi_m(x)} + \frac{1}{\Phi_m(x)} \int_{x_0}^{x} \underline{\varepsilon}(\xi) \Phi_{m+M}(\xi) d\xi \tag{33}$$

If we now take into consideration that $|\underline{\epsilon}(\xi)|$ according to (32) and $\lim_{\xi \to x_0} \underline{\epsilon}(\xi) = 0$ in the integration interval of x_0 to x certainly has a finite maximum $\hat{\epsilon}(x)$, there follows

$$\left|\frac{1}{\Phi_m(x)}\int_{x_0}^{x}\underline{\epsilon}(\xi)\Phi_{m+M}(\xi)\,d\xi\right| \leq \hat{\underline{\epsilon}}(x)\sum_{\mu=m}^{j(m+M)}\left|c_{n+M,\mu}\right|\left|\frac{\Phi_\mu(x)}{\Phi_m(x)}\right|$$

Hence we obtain, because of (3) and $\lim_{x \to x_0} \hat{\underline{\epsilon}}(x) = 0$ that the integral for $x \to x_0$ disappears on the right side of (33). Since, moreover, the remaining sum, on the basis of (3), also tends toward zero for $x \to x_0$, the proof of the theorem is completed.

Thus we may integrate asymptotic Φ-series always term by term. The inversion, in contrast, therefore the term-by-term differentiation, does not lead in every case to a correct result. For instance the function $F(x) = e^{-x}\sin(e^x)$ has, for $x \to \infty$, an asymptotic series of the form

$$0 + \frac{0}{x} + \frac{0}{x^2} + \;.....$$

Its derivative $F'(x) = -e^{-x}\sin(e^x) + \cos(e^x)$, however, oscillates for the approach of x to infinity and permits, therefore, certainly no such expansion.

In order to arrive at a general theorem also for the differentiation of Φ-series, one must always assume that not only the function $F(x)$ but also its derivative $F'(x)$ has an asymptotic Φ-expansion. Furthermore, we shall of course have to require that all derivatives $\Phi_n'(x)$ exist in J and again can be represented only as linear combinations of the functions $\Phi_n(x)$. These relations between the Φ_n' and the Φ_n are rather fixed by definition 4: If the considered Φ-sequence $\{\Phi_n\}_{J,x_0}$ is integrable, there follows for it from (28)

$$\Phi_n(x) = \sum_{\mu=n-M}^{j(n)} c_{n\mu}\Phi_\mu'(x) \qquad (n \geqq M, x \epsilon J) \qquad (34)$$

We shall here not investigate in more detail how far this infinite system of equations always has an inversion by which every derivative $\Phi_n'(x)$

is represented as a homogeneous linear combination of a finite number of $\Phi_\mu(x)$. However, insofar as such a representation

$$\Phi_n'(x) = \sum_{\substack{\text{Finite number}\\ \text{of } \mu}} d_{n\mu}\Phi_\mu(x) \qquad (n \geqq 0, x\epsilon J) \qquad (35)$$

exists for all n, it is, at first, certainly uniquely determined - because of the linear independence of the $\Phi_\mu(x)$ - and represents then moreover, for the same reason, also a solution of (34).

Likewise, we shall here not treat the question of what subscripts μ can appear in (35), but shall only agree upon the following:

Definition 5: An integrable Φ-sequence $\left\{\Phi_n\right\}_{J,x_0}$ is to be called differentiable, if, first, all its functions $\Phi_n(x)$ in the entire J are continuously differentiable and, second, the infinite system of equations (34) following from (28) has an inversion of the form (35) which is then, of course, uniquely determined.

Evidently our three examples (4), (5), and (6) all represent differentiable Φ-sequences. For (4) and (5) this is perfectly clear, for (6) it follows from

$$\frac{d}{dx}\left(x^k \ln^{k-l} x\right) = kx^{k-1}\ln^{k-l}x + (k-l)x^{k-1}\ln^{k-l}x$$

Now we can show:

Theorem 7: The function $F(x)$, continuously differentiable in J may be represented for $x \to x_0$ by the asymptotic Φ-expansion

$$F(x) \sim \sum_{n=0}^{\infty} a_n\Phi_n(x) \qquad (36)$$

where the Φ-sequence $\left\{\Phi_n\right\}_{J,x_0}$ is differentiable in J. If then $F'(x)$ also has, for $x \to x_0$, an asymptotic Φ-series in J, the latter results from (36) by term-by-term differentiation and subsequent reordering according to the Φ_n.

Proof: Let the Φ-series of $F'(x)$ existing according to assumption read, for instance

$$F'(x) \sim \sum_{n=0}^{\infty} b_n \Phi_n(x) \tag{37}$$

We assume first that x_0 is finite. Then there follows by integration of (37) according to theorem 6

$$F(x) = a_0 + \int_{x_0}^{x} F'(\xi)d\xi \sim (a_0 + B_0) + \sum_{n=1}^{\infty} B_n \Phi_n(x) \tag{38}$$

with the coefficients B_n originating by term-by-term integration of (37) and subsequent reordering; for simplification, we put $\Phi_0(x) \equiv 1$. However, since - because of theorem 1 - a function uniquely determines its asymptotic series, there must, on the other hand,

$$B_0 = 0, \quad B_n = a_n \qquad (n \geqq 1)$$

be valid. That is, we obtain from the expansion (37) of $F'(x)$ by term-by-term integration and subsequent reordering precisely the Φ-series (36) of $F(x)$. On the basis of the differentiability of the Φ-sequence according to definition 5, this process may be directly performed also in the inverse direction. If, therefore, (36) is differentiated term by term and subsequently reordered, (37) must, again, always be the result. This precisely was the assertion.

Let now x_0 be $\pm\infty$. Then one has with two finite values $x, x_1 \epsilon J$

$$F(x) = c_1 + \int_{x_1}^{x} F'(\xi)d\xi$$

$$= c_1 + \sum_{n=0}^{M-1} b_n \int_{x_1}^{x} \Phi_n(\xi)d\xi + \int_{x_1}^{x} \left[F(\xi) - \sum_{n=0}^{M-1} b_n \Phi_n(\xi)\right] d\xi$$

or

$$F(x) = c_1 + \sum_{n=0}^{M-1} b_n \Psi_n(x) + \int_{x_0=\pm\infty}^{x} \left[F(\xi) - \sum_{n=0}^{M-1} b_n \Phi_n(\xi)\right] d\xi \tag{39}$$

where for abbreviation,

$$\psi_n(x) = \int_{x_1}^{x} \Phi_n(\xi)\,d\xi$$

On the basis of the assumption regarding M, these integrals are at any rate divergent in the limiting process $x_1 \to x_0 = \pm\infty$, that is, the boundary values $\lim\limits_{x \to x_0} \psi_n(x)$ do not exist. On the other hand, there follows with the aid of theorem 2, because in (39) $F(x)$ as well as the integral on the right side possess asymptotic expansions for $x \to x_0$ that $\sum\limits_{n=0}^{M-1} b_n\psi_n(x)$ also may be represented by such a series for $x \to x_0$. This is a contradiction, unless $b_0 = b_1 = \ldots = b_{M-1} = 0$, that is, $\sum\limits_{n=0}^{M-1} b_n\psi_n(x) \equiv 0$ is valid. Now, however, one may draw a conclusion again in exactly the same manner as in the case of the finite x_0 whereby the theorem may be regarded as proved.

REFERENCES

1. Schlichting, H.: Grenzschicht-Theorie, Verlag G. Braun (Karlsruhe), 1951.

2. Watson, E. J.: The Asymptotic Theory of Boundary Layer Flow with Suction. A.R.C., R & M No. 2619, Sept. 1947.

3. Iglisch, R.: Exakte Berechnung der laminaren Grenzschicht an der längsangeströmten ebenen Platte mit homogener Absaugung. Schriften d. Deutschen Akad. d. Luftfahrtforschung, Bd. 8 B, Heft 1, 1944.

4. Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verh. d. III. Intern. Math. Kongr. (Heidelberg), 1904, Verlag Teubner (Leipzig), 1905.

5. Lew, H. G., and Mathieu, R. D.: Boundary Layer Control by Porous Suction. The Penn. State Univ., Dept. of Aerod. Engineering. Tech. Rep. No. 3, 1954.

6. Jones, Sir M., and Head, M.: The Reduction of Drag by Distributed Suction. Third Anglo-American Aeronautical Conference (Brighton), 1951. (Published by the Royal Aero. Soc. (London), 1952.)

7. Pfenninger, W.: Investigations on Reductions of Friction on Wings, in Particular by Means of Boundary Layer Suction. NACA TM 1181, 1947.

8. Von Doenhoff, A. E., and Loftin, L. K., Jr.: Present Status of Research on Boundary Layer Control. NACA RM L8J29, 1949.

9. Rheinboldt, W.: Über die äussere Randbedingung bei den Grenzschichtgleichungen. Festband "50 Jahre Grenzschichtforschung" Vieweg-Verlag, (Braunschweig), erscheint im Sommer 1955.

10. Görtler, H.: Ein Differenzenverfahren zur Berechnung laminarer Grenzschichten. Ing. - Arch. 16, 1948, p. 173.

11. Courant, R., and Hilbert, D.: Methoden der mathematischen Physik. (1.Aufl.) Springer Verlag (Berlin), 1937.

12. Goldstein, S.: Concerning some Solutions of the Boundary Layer Equations in Hydrodynamics. Proc. Cambridge Phil. Soc. 26, 1930, p. 19.

13. Whittaker, E. T., and Watson, G. N.: A Course of Modern Analysis. Fourth ed., The Univ. Press (Cambridge), 1952.

L-401

14. Knopp, K.: Theorie und Anwendung der unendlichen Reihen. (4.Aufl.) Springer Verlag (Berlin und Heidelberg), 1947.

15. Borel, É.: Lecons sur les Séries Divergentes. Verlag Gauthier-Villars (Paris), 1901.

16. Goldstein, S.: On Laminar Boundary Layer Flow near a Position of Separation. Quart, Jour. of Mech. and Appl. Math. Bd. 1, 1948, p. 43.

17. Görtler, H.: Weiterentwicklung eines Grenzschichtprofils bei gegebenem Druckverlauf. Z.a.M.M., 19, 1939, p. 129.

18. Ulrich, A.: Die ebene laminare Reibungsschicht an einem Zylinder. Arch. d. Math. 2, 1949, p. 33.

19. Witting, H.: Verbesserung des Differenzenverfahrens von H. Görtler zur Berechnung laminarer Grenzschichten. Dissertation Freiburg/Br. 1953 abgedruckt in Z.A.M.P., Bd. IV, p. 376.

20. Witting, H.: Über die Instabilitäten der Prandtlschen Grenzschichtgleichungen. Festband "50 Jahre Grenzschichtforschung" Vieweg-Verlag (Braunschweig), erscheint im Sommer 1955.

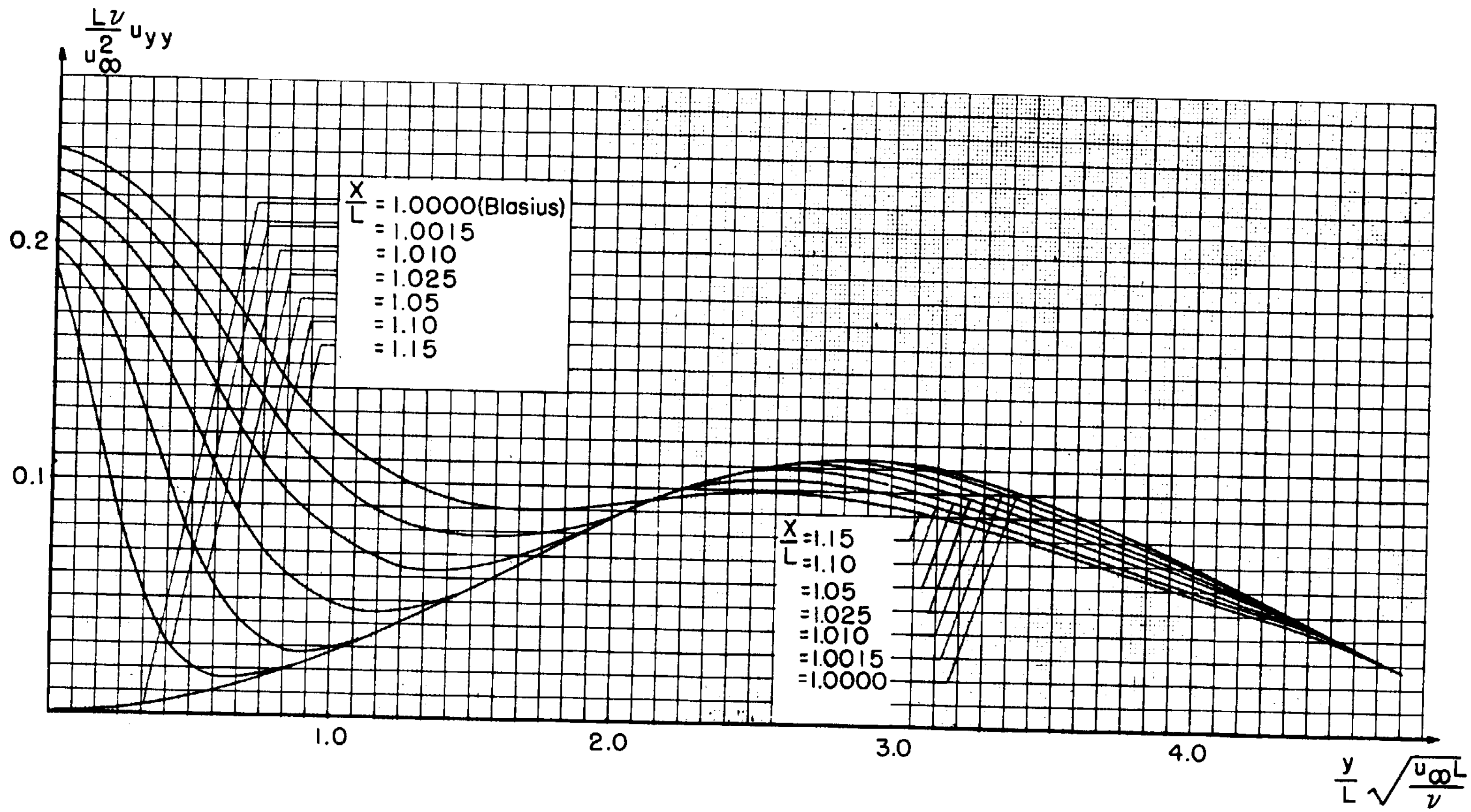

Figure 1. - The second derivatives $u^*_{y^*y^*}$ at several points x^* for the example of the plane plate with $v_o^* = 0.5$.

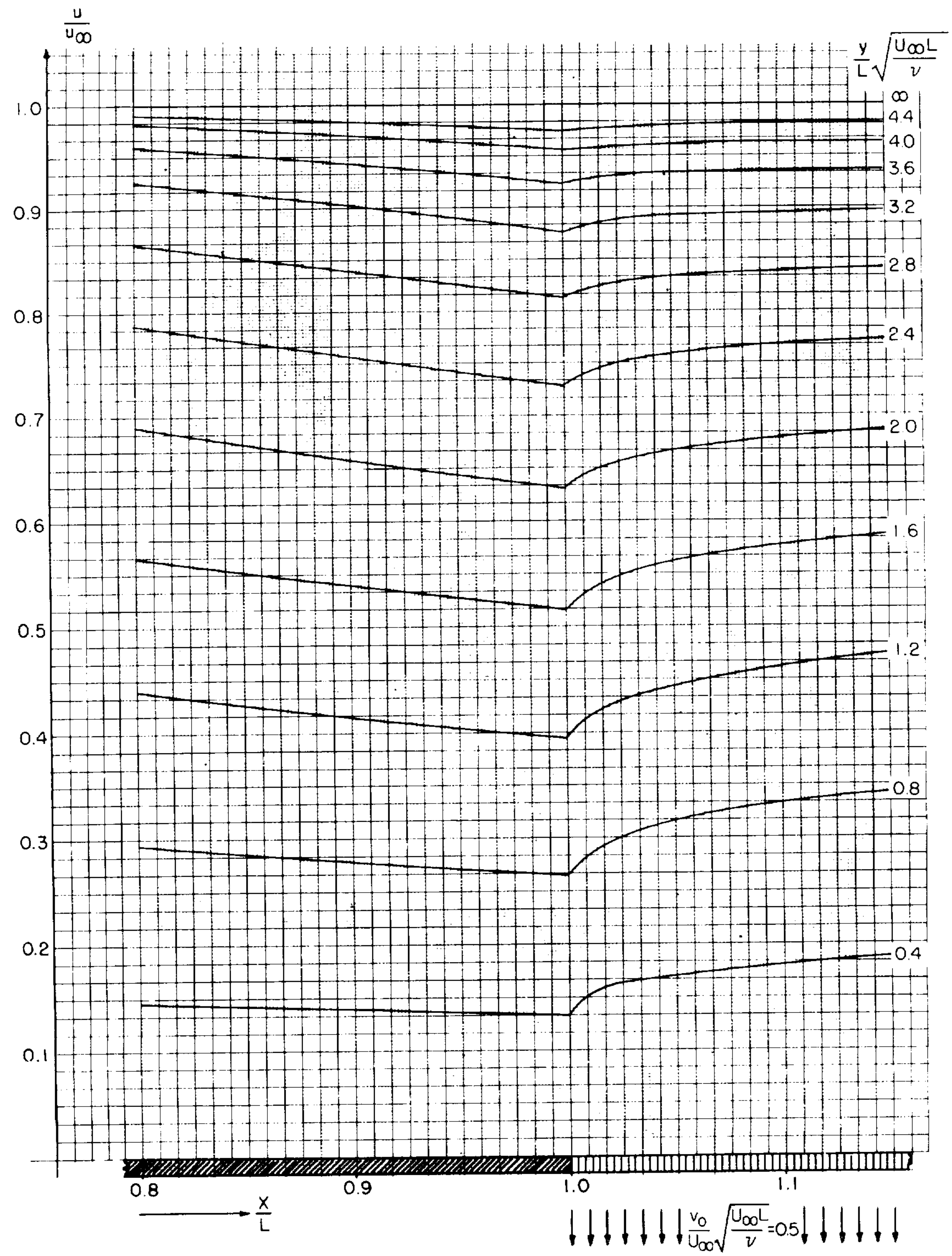

Figure 2. - Velocity distribution for constant distance from the wall for the plane plate with $v_0^* = 0.5$.

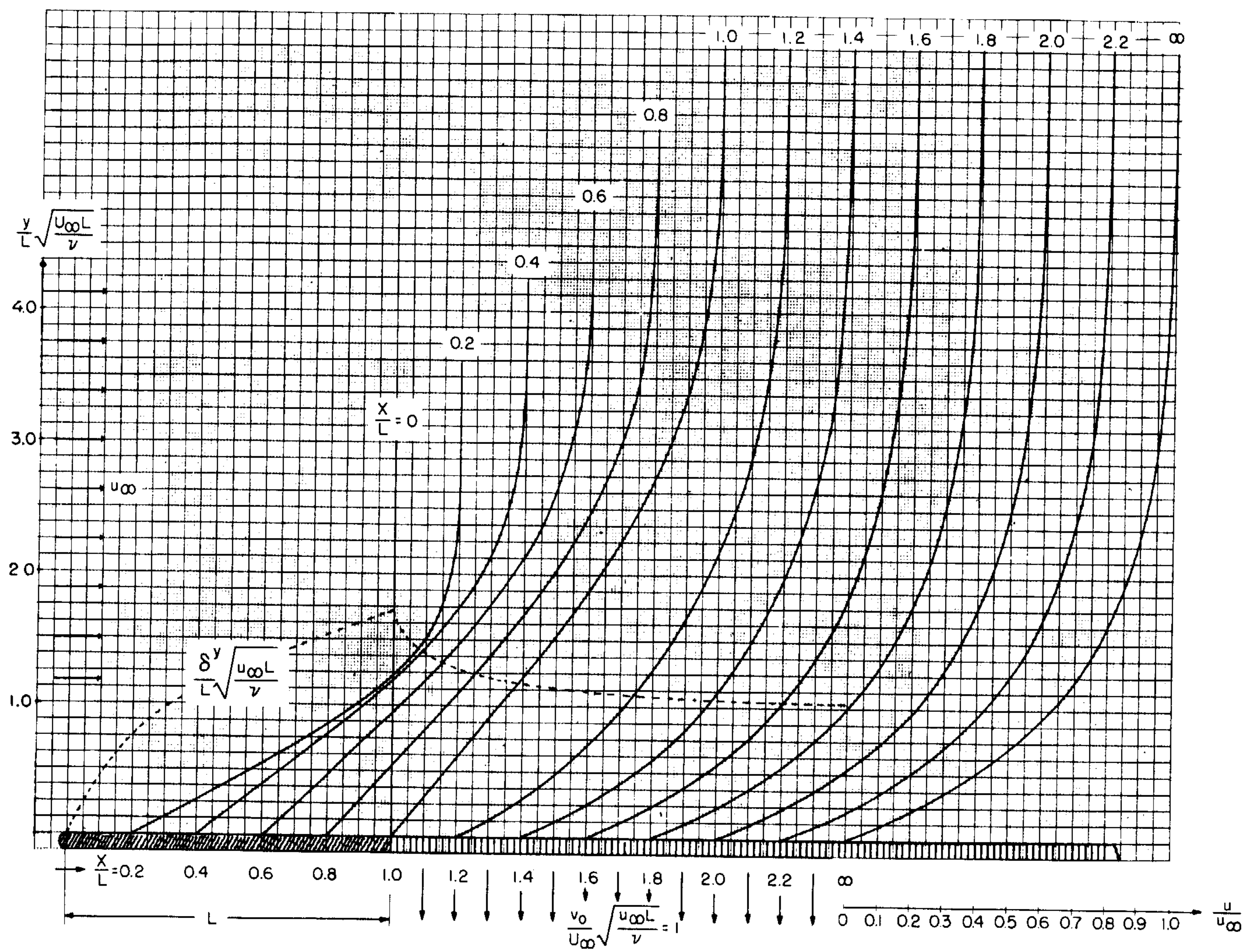

Figure 3. - A few calculated velocity profiles for the plane plate with $v_0^* = 1.0$. The considerable variation of the profile form caused by the suction is noteworthy.

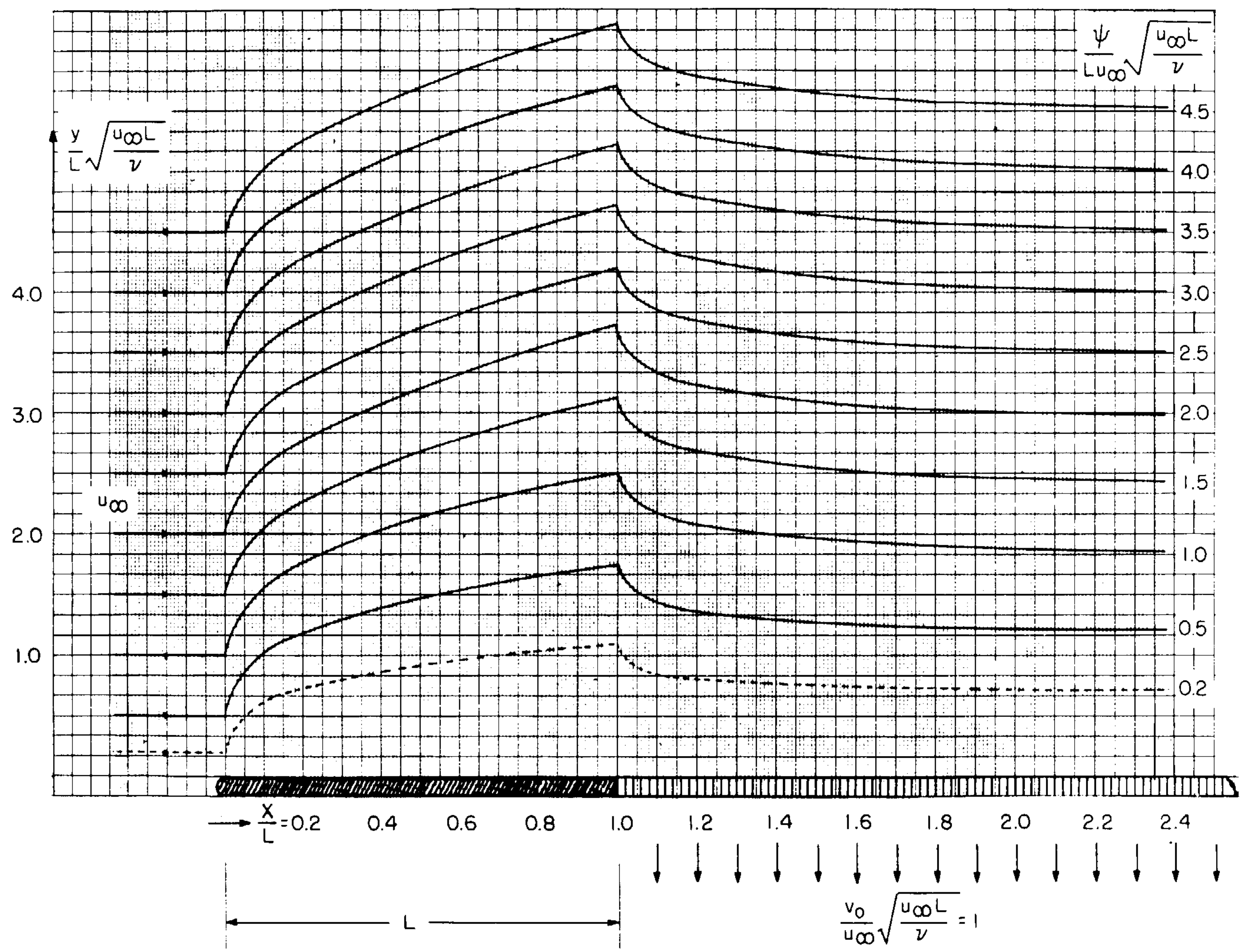

Figure 4.- Streamline pattern for the plane plate with $v_0^* = 1.0$. (The four following figures serve to illustrate the differences for various suction values.)

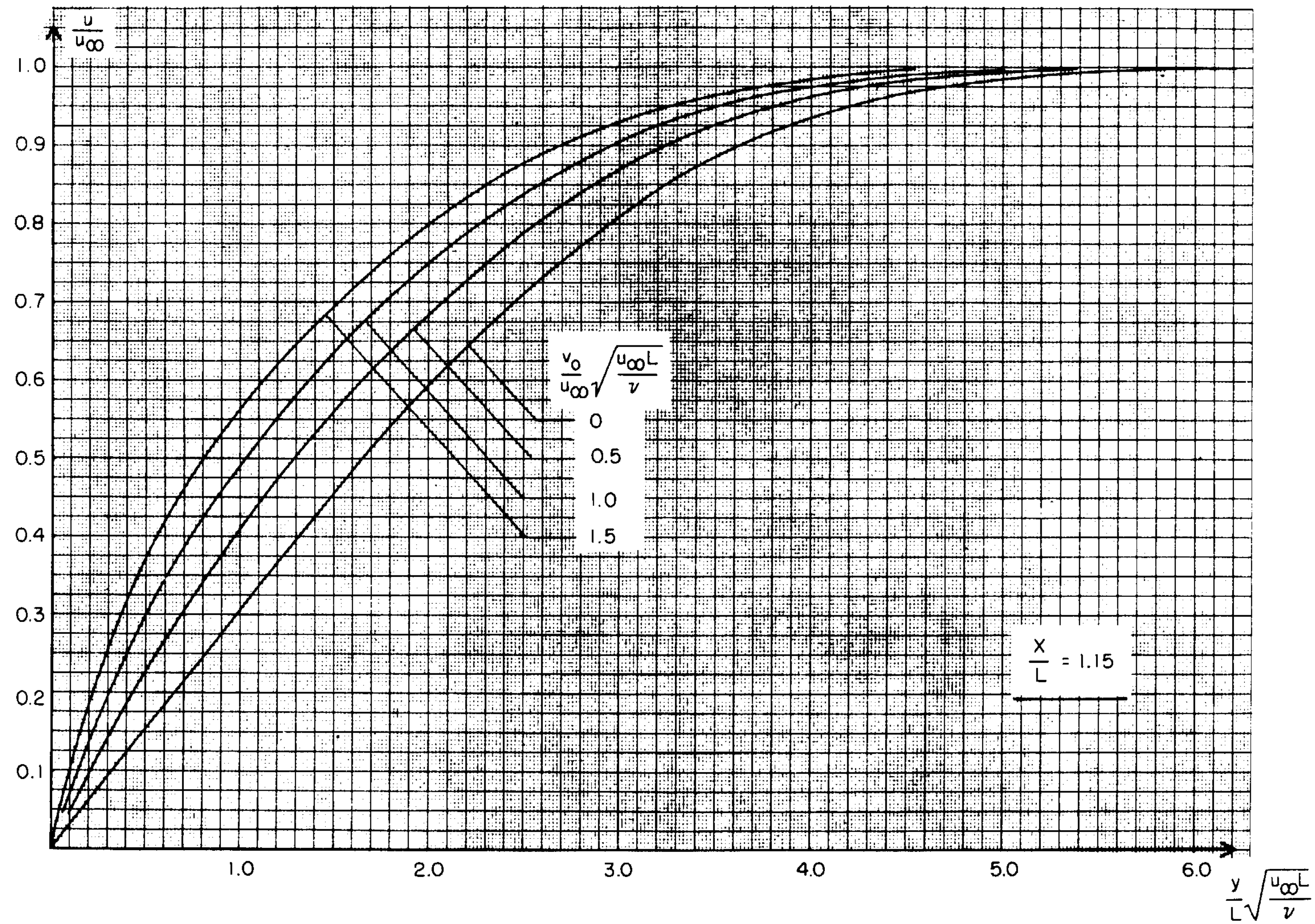

Figure 5. - Comparison of the velocity profiles for various suction values at the point $x^* = 1.15$.

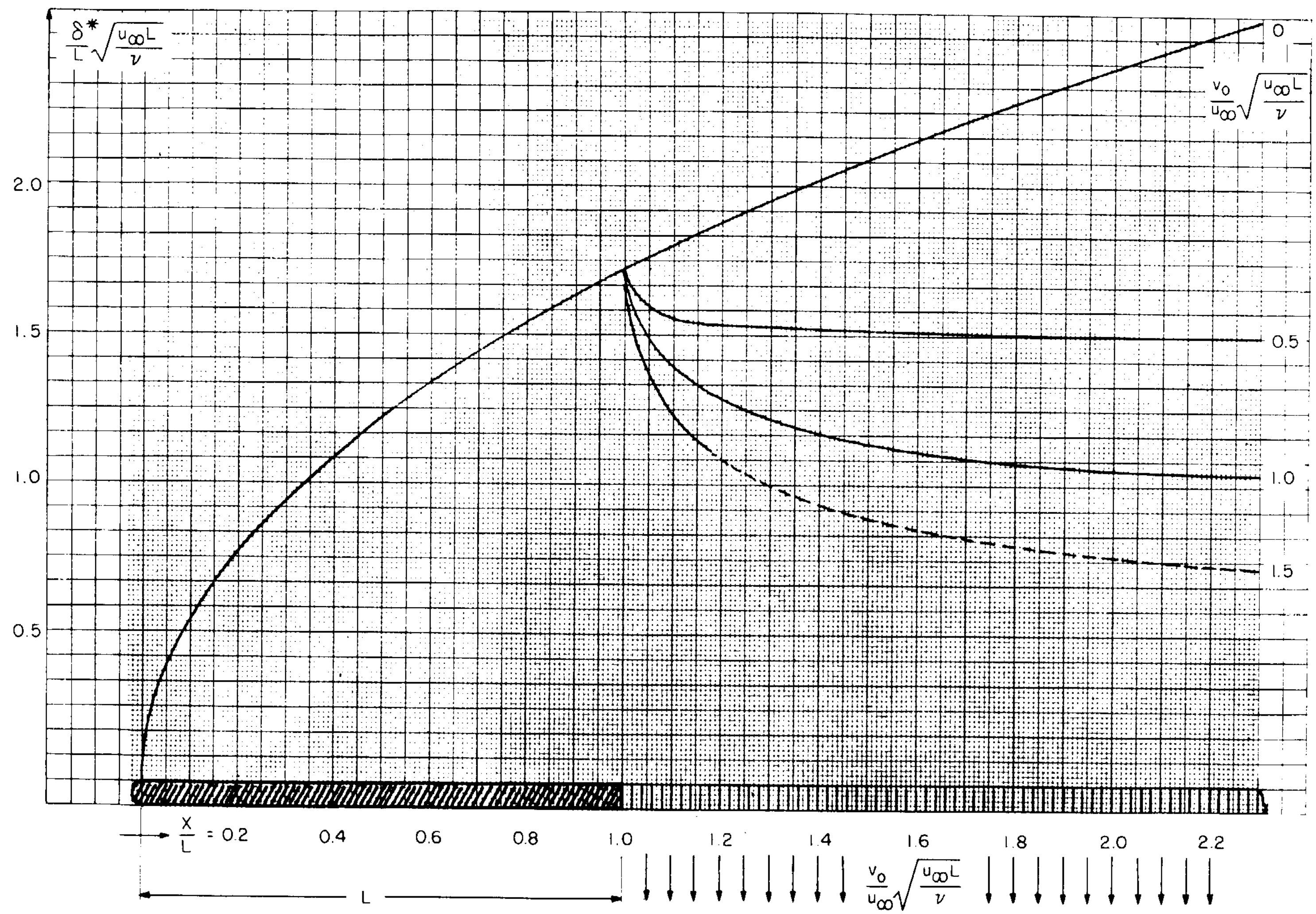

Figure 6. - Curves of the displacement thickness δ^* for the plane plate for various suction values.

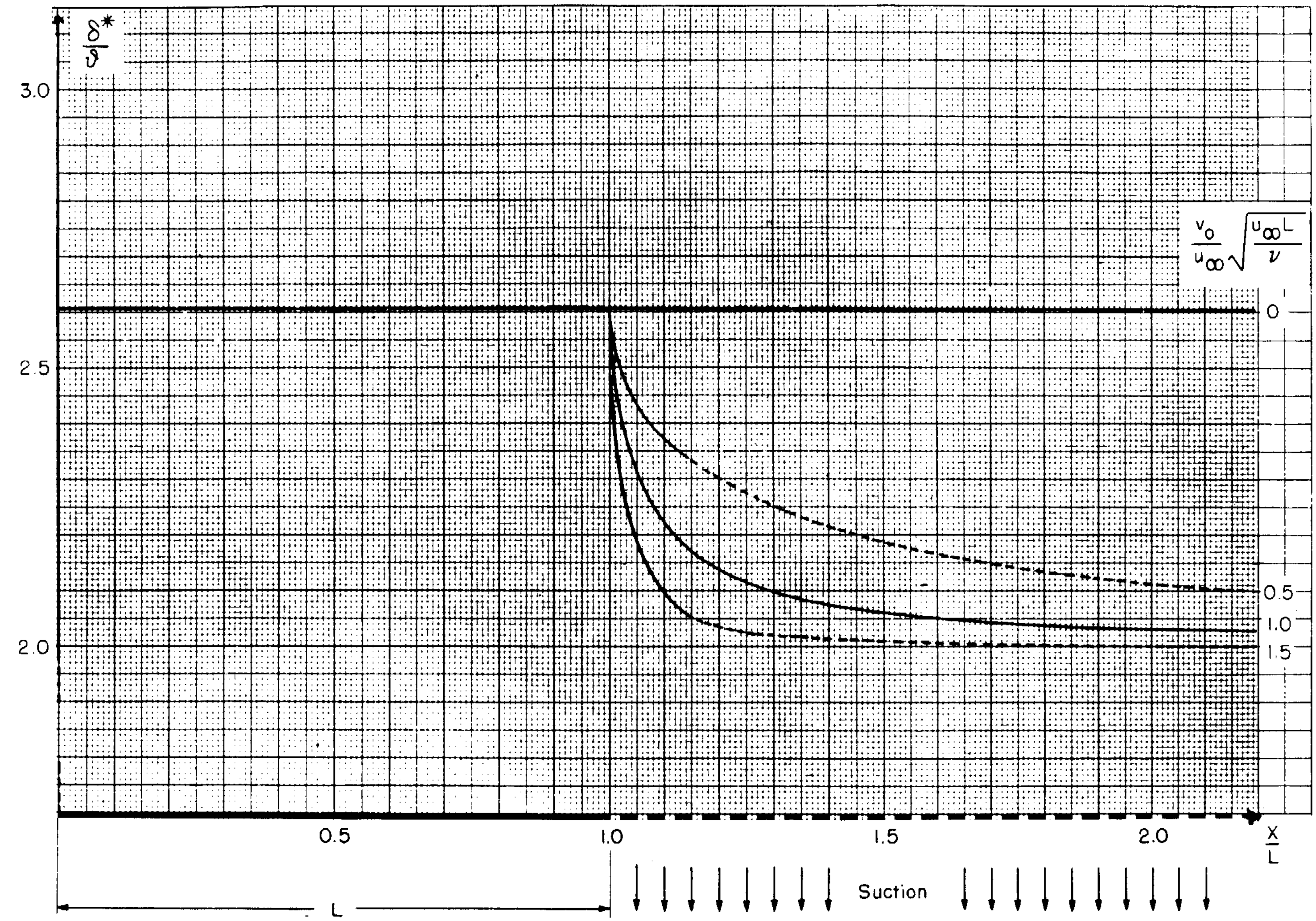

Figure 7. - Comparison of the profile parameter $\frac{\delta^*}{\vartheta}$ for various suction values.

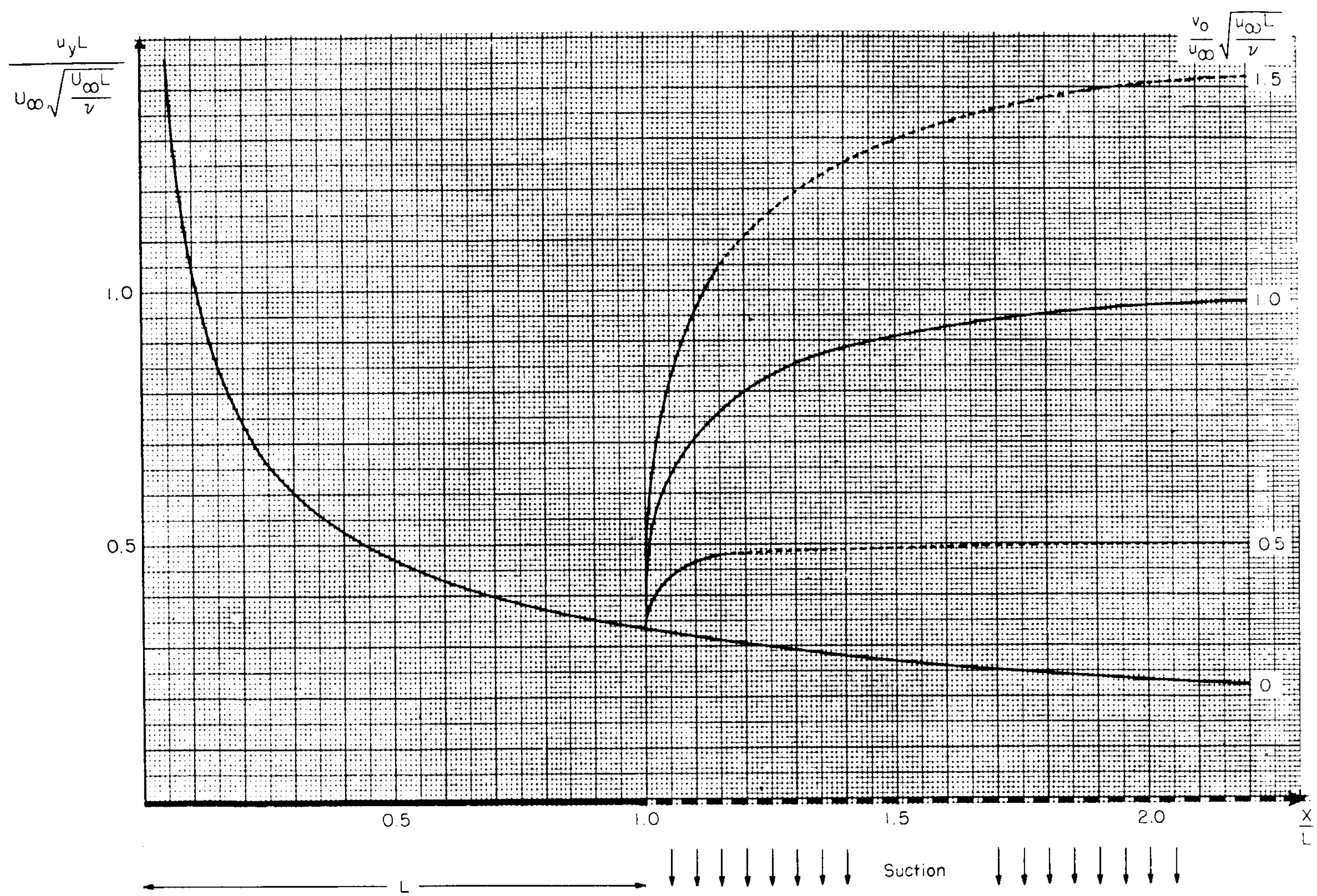

Figure 8. - Curves of the wall shear stress $u^*_{y^*}(x^*, 0)$ for various suction values.

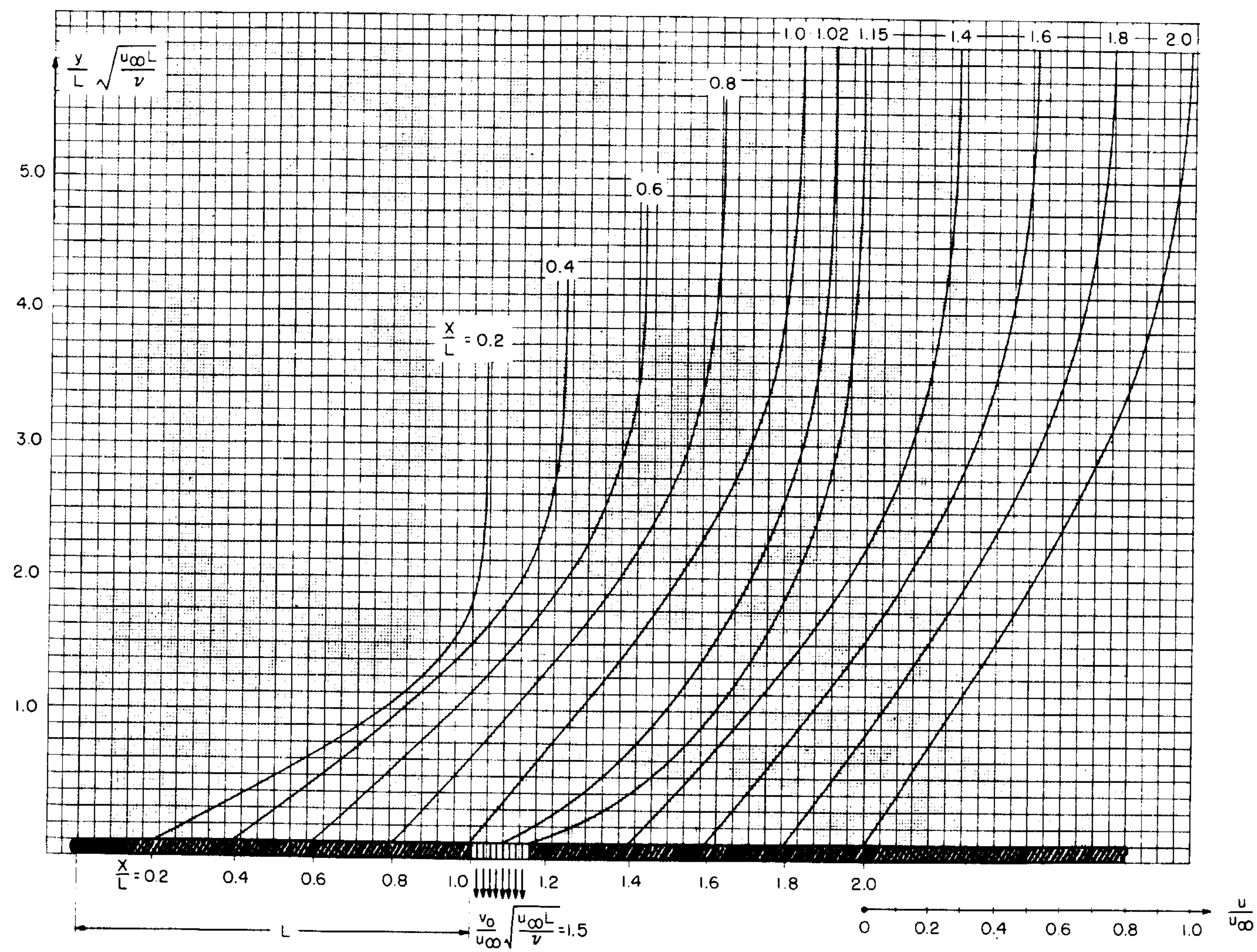

Figure 9. - Several calculated velocity profiles for the plane plate with suction $v_0^* = 1.5$ in the range $1.0 \leqq x \leqq 1.15$.

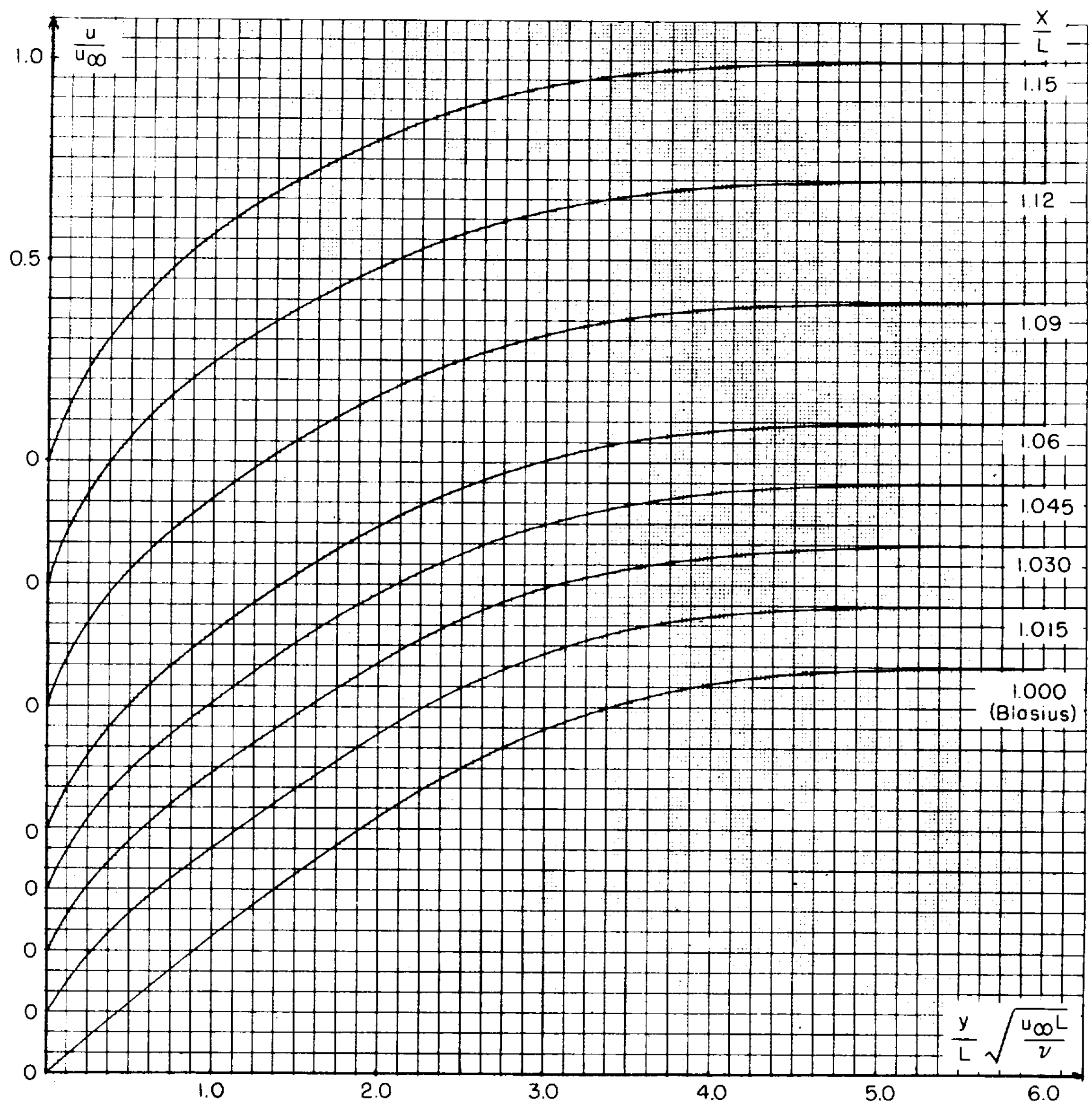

Figure 9(a). - Section from figure 9: A few velocity profiles within the region of suction.

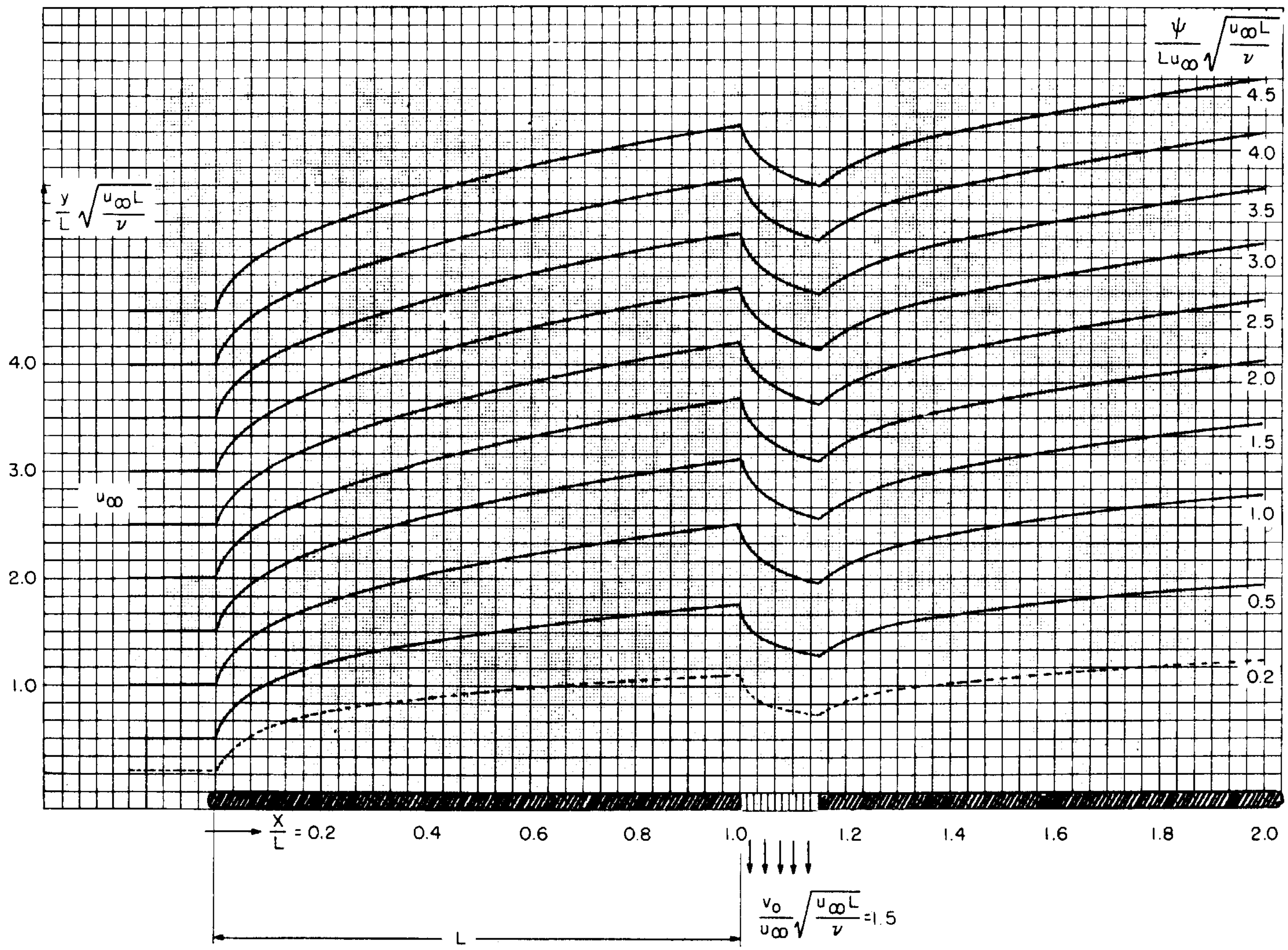

Figure 10.- Streamline pattern for the plane plate with suction $v_0^* = 1.5$ in the range $1.0 \leq x \leq 1.15$.

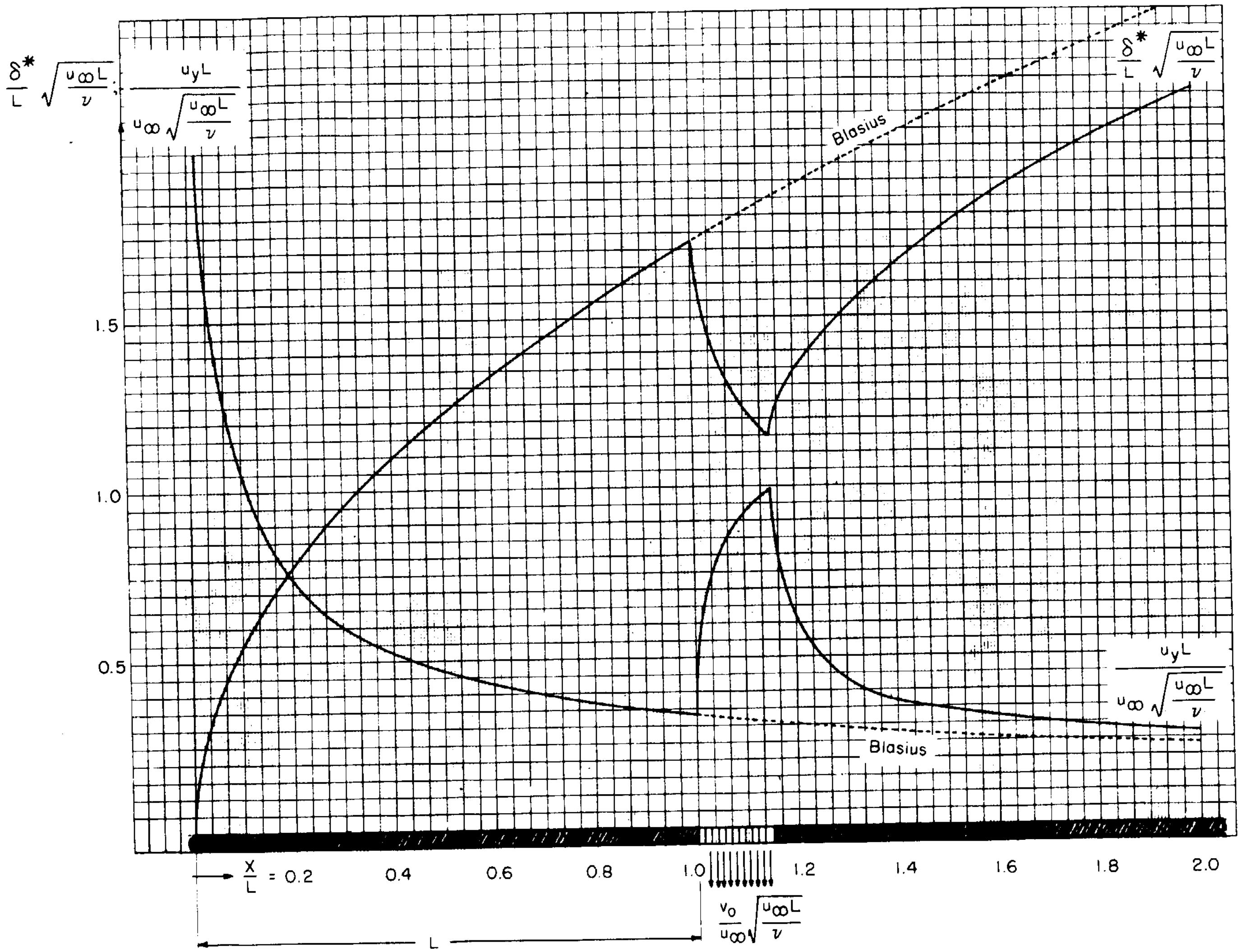

Figure 11. - Displacement thickness and wall shear stress for the plane plate with suction $v_0^* = 1.5$ in the range $1.0 \leq x \leq 1.15$.

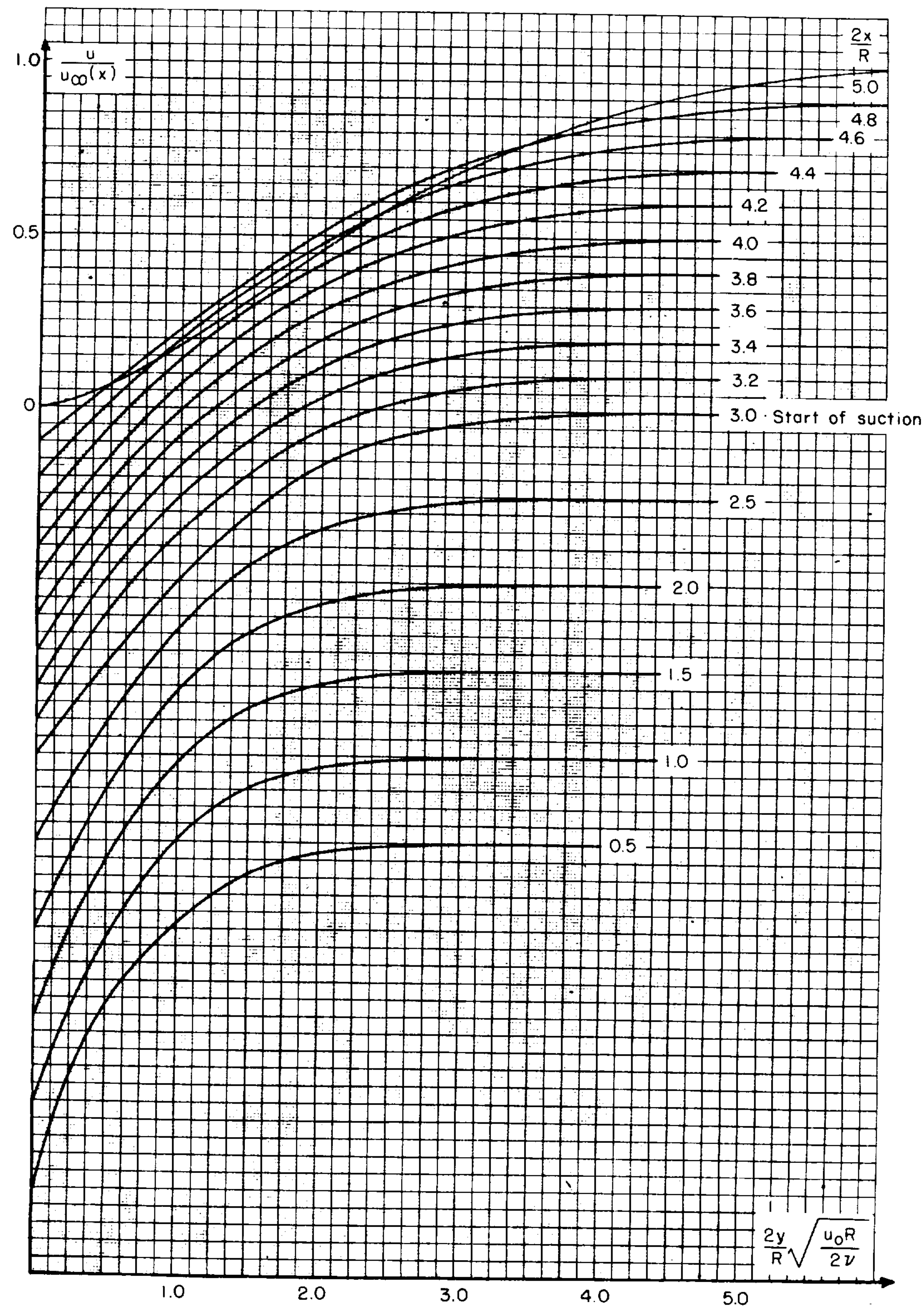

Figure 12. - A few calculated velocity profiles for the circular cylinder with suction.

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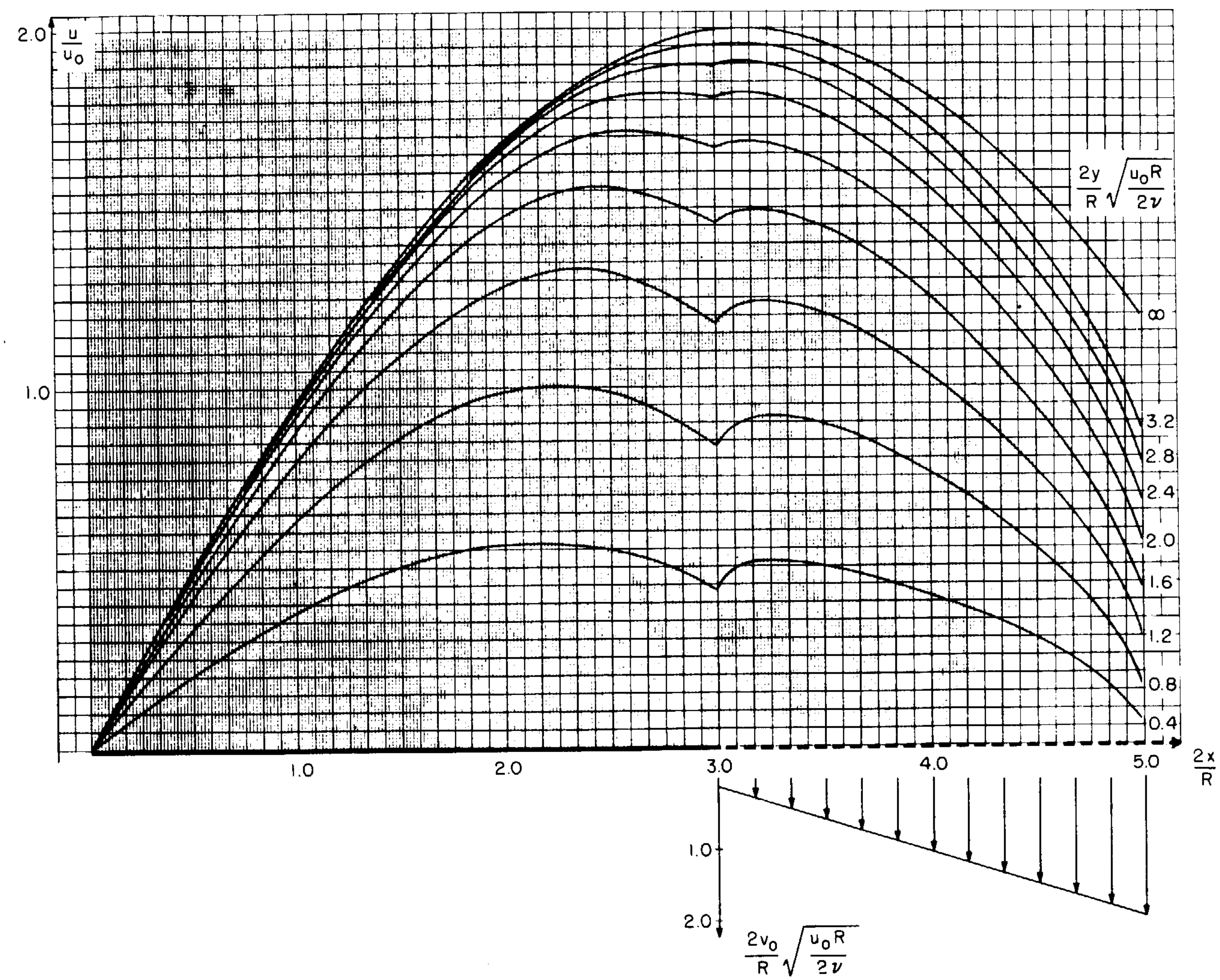

Figure 13. - Velocity distribution for constant distance from the wall for the circular cylinder with suction.

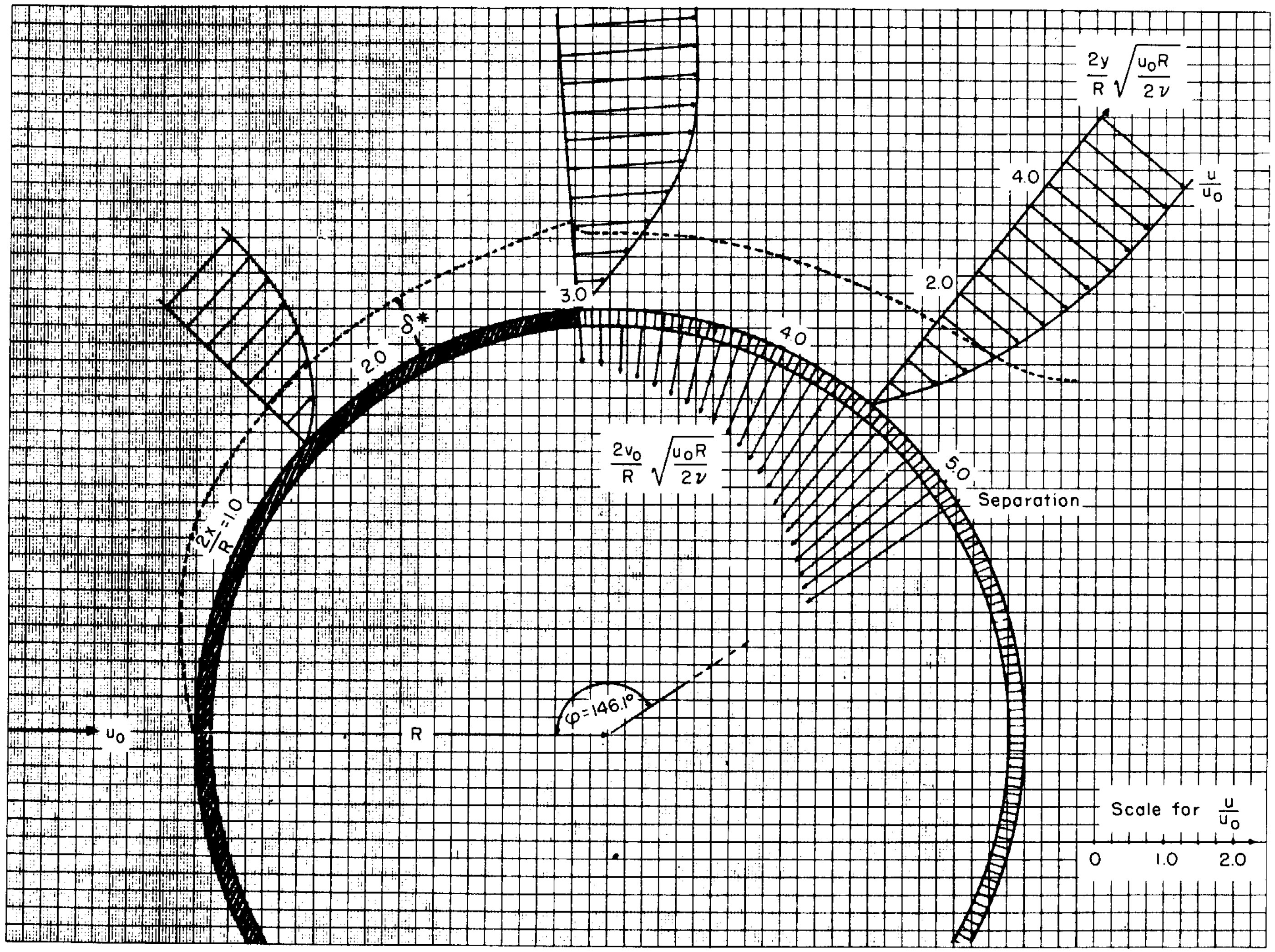

Figure 14. - Displacement thickness and three velocity profiles for the circular cylinder with suction.

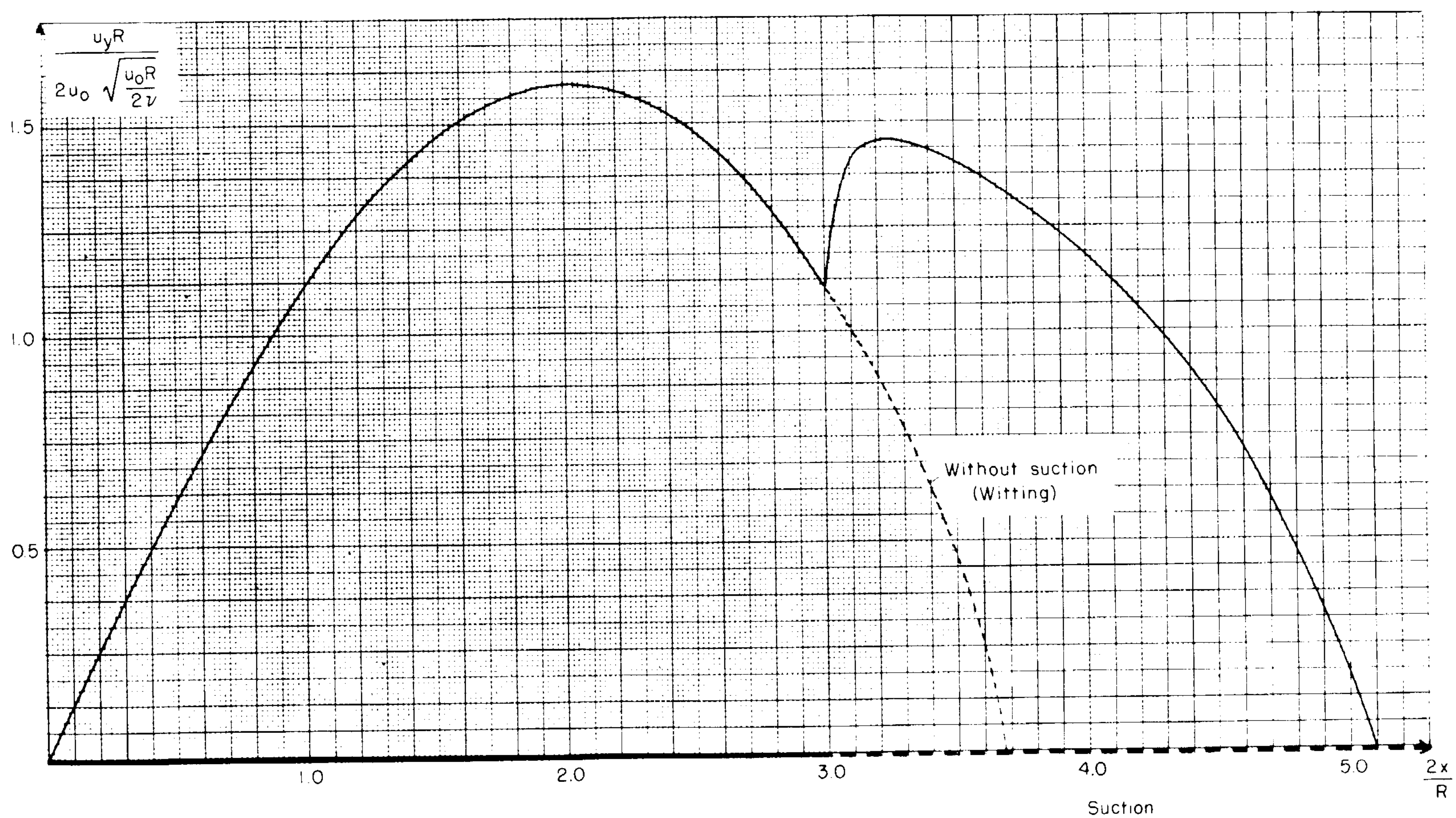

Figure 15.- Comparison of the wall shear stress for the circular cylinder with and without suction.

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